O. A. Ladyzhenskaya and N. N. Ural' tseva

The equation

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Abstract

Full Text

O. A. Ladyzhenskaya and N. N. Ural' tseva

On the Differential Properties of Bounded Generalized Solutions of Multidimensional Quasilinear Elliptic Equations and Variational Problems

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The equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(x,u,u_x)) + a(x,u,u_x) = 0 \tag{1}$$

is investigated.

We shall assume that $m > 1$; a_i and a are measurable functions of their arguments satisfying the conditions

$$|a_i(x,u,p_i)|\, p + |a(x,u,p_i)| \leq \mu(|u|)(1+p)^m,$$

$$a_i(x,u,p_i)p_i \geq \nu(|u|)p^m - \mu(|u|), \tag{2}$$

where $p = \left[\sum_j p_j^2\right]^{1/2}$. In a number of cases we shall assume that a_i and a, in addition to condition (2), satisfy condition (3), namely, that they have first differentials with respect to all their arguments and

$$\nu(|u|)(1+p)^{m-2}\sum_i \xi_i^2 \leq \frac{\partial a_i(x,u,p_j)}{\partial p_j}\,\xi_i\xi_j \leq \mu(|u|)(1+p)^{m-2}\sum_i \xi_i^2,$$

$$\left|\frac{\partial a_i}{\partial p_j}\right| p^2 + \left|\frac{\partial a_i}{\partial u}\right| p + \left|\frac{\partial a}{\partial p_i}\right| p + \left|\frac{\partial a}{\partial u}\right| \leq \mu(|u|)(1+p)^m. \tag{3}$$

Here and below $\nu(t)$ and $\mu(t)$ are, respectively, monotonically nonincreasing and nondecreasing positive functions for $t \geq 0$.

We shall call a function $u(x)$ from $W_m^1(\Omega)^*$ a generalized solution of equation (1) if it satisfies the identity

$$I(u,\eta) \equiv \int_\Omega [a_i(x,u,u_x)\eta_{x_i} - a(x,u,u_x)\eta]\, dx = 0 \tag{4}$$

for every bounded η from $\mathring{W}^1_m(\Omega)$.

We have proved $(^{1,2})$ that every bounded generalized solution $u(x)$ of equation (1) belongs to $C_{0,\alpha}(\Omega)$, with some $\alpha > 0$ depending only on $\mu(\max_\Omega |u|)$ and $\nu(\max_\Omega |u|)$ occurring in (2).

Lemma 1. *For a bounded generalized solution* $u(x)$ *of equation (1), the inequalities*

$$\int_{K(\rho)} |\nabla u|^m \, dx \leq c\rho^{n-m+\alpha}, \tag{5}$$

$$\int_{K(\rho)} |x-y|^{-n+m-\alpha/2} |\nabla u|^m dx \leq c\rho^{\alpha/2}, \tag{6}$$

where $K(\rho)$ is any ball of radius ρ belonging to Ω, and the constant c depends only on $\mu(\max_\Omega |u|)$ and $\nu(\max_\Omega |u|)$, appearing in (2).

For the proof one must take in (4) $\eta(x) = [u(x) - u(x_0)]\zeta^m(x)$, where x_0 is the center of the ball $K(\rho)$, and $\zeta(x)$ is a smooth, nonnegative cut-off function equal to zero outside $K(\rho)$, and use (2) and $|u(x) - u(x_0)| \leq c\rho^\alpha$. This leads to (5). Inequality (6), as is known, follows from (5) (see, for example, $(^4)$).

Lemma 2. For any bounded generalized solution $u(x)$ of equation (1), for $m \geq 2$, the inequality

$$\int_{K(\rho)} (1 + |\nabla u|)^m \xi^2 dx \leq c\rho^\alpha \int_{K(\rho)} (1 + |\nabla u|)^{m-2} |\nabla \xi|^2 dx \tag{7}$$

holds for any bounded ξ from $\overset{0}{W}{}^1_m(K(\rho))$, with a constant c depending only on $\mu(\max_\Omega |u|)$ and $\nu(\max_\Omega |u|)$, appearing in (2).

For the proof one must take in (4), as η, the function $[u(x) - u(x_0)]\xi^2(x)$ and use, along with (2), the well-known inequality

$$\int_{K(\rho)} \xi^2 dx \leq c\rho^2 \int_{K(\rho)} |\nabla \xi|^2 dx,$$

valid for any $\xi \in \overset{0}{W}{}^1_2(\Omega)$.

Lemma 2′. If $b(x) > 0$ and, for any $\rho > 0$ and $y \in \Omega$, the integrals

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[*] For the definition of $W^1_m(\Omega)$ and $\mathring{W}^1_m(\Omega)$, see, for example, $(^3)$.

$$\int_{K(\rho)} |x-y|^{-n+m-\alpha/2} b^m(x)dx \le c_1\rho^{\alpha/2}, \quad \alpha > 0, \quad 1 < m \le 2,$$

then

$$\int_{K(\rho)} b^m \xi^2 dx \le c\rho^{2\alpha/m} \int_{K(\rho)} b^{m-2} |\nabla\xi|^2 dx \tag{8}$$

for any bounded ξ from $\overset{0}{W}{}^1_m(K(\rho))$, with a constant c depending only on c_1, α, m.

To prove (8) we use the fact that, for any $\xi \in \overset{0}{W}{}^1_m(K(\rho))$, almost everywhere the identity

$$\xi(x) = \frac{1}{\varkappa_n} \int_{K(\rho)} \frac{\partial\xi(y)}{\partial y_i} \frac{\partial}{\partial y_i} \frac{1}{|x-y|^{n-2}}\, dy$$

holds, where $\dfrac{\varkappa_n}{n-2}$ is the surface area of the unit sphere in n-dimensional space. In view of this and of the conditions of the lemma,

$$\begin{aligned}
\int_{K(\rho)} b^m\xi^2 dx &\le c\int b^m(x)\left(\int |x-y|^{1-n}|\nabla\xi|\, dy\right)^2 dx \\
&\le c\int b^m(x)\left(\int |x-y|^{-n+m-\alpha/2}|\nabla\xi(y)|^2 b^{m-2}(y)\, dy\right) \\
&\quad \times \left(\int |x-y|^{-n-m+2+\alpha/2} b^{2-m}(y)\, dy\right) dx \\
&\le c_1\rho^{2\alpha/m-\alpha/2}\int b^m(x)\left(\int |x-y|^{-n+m-\alpha/2}|\nabla\xi(y)|^2 b^{m-2}(y)\, dy\right) dx \\
&\le c_2\rho^{2\alpha/m}\int b^{m-2}(y)|\nabla\xi(y)|^2 dy.
\end{aligned}$$

It follows from Lemma $2'$ that Lemma 2 is also valid for $1 < m \le 2$.

Theorem 1. For a bounded generalized solution $u(x)$ of equation (1), a uniqueness theorem in the small is valid. More precisely: two bounded-

of generalized solutions $u'(x)$ and $u''(x)$, equal to each other on the surface of the ball $K(\rho)$, coincide in $K(\rho)$, provided only that the radius ρ is less than a certain number determined by $\mu\left(\max\limits_{\Omega}|u'|, |u''|\right)$ and $\nu\left(\max\limits_{\Omega}|u'|, |u''|\right)$, entering into (2) and (3).

For any bounded $\eta(x) \in \overset{\circ}{W}{}^1_m(K(\rho))$ we have

$$0 = I(u', \eta) - I(u'', \eta) = \int_0^1 \frac{d}{dt} I(u^t, \eta)\, dt,$$

where $u^t = (1-t)u'' + tu'$. Differentiating with respect to t and putting $\eta = u' - u''$, we arrive at an equality from which, using (3), we obtain

$$\int_{K(\rho)} (1 + |\nabla u'| + |\nabla u''|)^{m-2} |\nabla \eta|^2 dx \leq c \int_{K(\rho)} (1 + |\nabla u'| + |\nabla u''|)^m \eta^2 dx. \quad (9)$$

Hence, on the basis of Lemma 2 for $m \geq 2$ and Lemma 2′ for $1 < m \leq 2$, we conclude that $\eta(x)$ is identically zero.

Theorem 2. If conditions (2) and (3) are satisfied, then any bounded generalized solution $u(x)$ of equation (1) has generalized second derivatives and satisfies equation (1) almost everywhere. For it,

$$\int_{\Omega'} \left[|\nabla u|^{m+2} + (1 + |\nabla u|)^{m-2} \sum_{i,j} u^2_{x_i x_j} \right] dx < c \subset \Omega', \quad (10)$$

where Ω' is an arbitrary strictly interior subdomain of Ω. If S and $\varphi = u|_S$ are twice continuously differentiable, then (10) is also true for $\Omega' = \Omega$.

Take in (4) $\eta(x) = \left(u_{(k)}(x)\xi^2(x)\right)_{(\bar{k})}$, where $\xi(x)$ is a twice continuously differentiable nonnegative cutoff function, equal to zero outside $K(\rho)$, and the symbols $(\ )_{(k)}$ and $(\ )_{(\bar{k})}$ denote right and left difference quotients with respect to x_k. After this we transfer $(\ )_{(\bar{k})}$ to the first factor and carry out estimates, using (3). This will lead us to the inequality

$$\int_\Omega \sum_i B^{m-2} u^2_{x_i(k)} \xi^2 dx \leq c \int_\Omega \left[B^{m-2} u^2_{(k)} |\nabla \xi|^2 + B^m u^2_{(k)} \xi^2 \right] dx, \quad (11)$$

where $B(x) = 1 + |\nabla u(x)| + |\nabla u(x + \Delta x_k)|$. Since $u(x + \Delta x_k)$ is, just as $u(x)$, a bounded generalized solution of an equation of type (1), Lemmas 2 and 2′ are valid for them. From these lemmas and (11) there follows the uniform boundedness of the integrals

$$\int_\Omega \left(B^{m-2} \sum_i u^2_{x_i(k)} + B^m u^2_{(k)} \right) \xi^2 dx < c,$$

on the basis of which we are convinced of the validity of all assertions of the theorem for $\Omega' \subset \Omega$. The proof of (10) for $\Omega' = \Omega$ is carried out in the same way as in (⁵), by straightening the boundary and reducing the boundary values to zero.

The facts established here for equations (1) make it possible to strengthen the results proved by us earlier (¹,²) concerning bounded generalized solutions of the variational problem of determining the minimum of the functional

$$J(u) = \int_\Omega F(x, u, u_x)\, dx_1 dx_n, \qquad u|_S = \varphi. \tag{12}$$

The Euler equation for J is a special case of equations of the form (1), and for it conditions (2), (3) follow from the so-called "natural" assumptions on F. Namely, from papers (¹,²) and the propositions proved here there follows:

Theorem 3. Any bounded function $u(x)$ from $W_m^1(\Omega)$, for which

$$\delta J(u) = \int_\Omega (F_{u_{x_i}}(x, u, u_x)\, \eta_{x_i} + F_u \eta)\, dx = 0$$

for every bounded

$$\eta(x) \in \overset{\circ}{W}{}_m^1(\Omega),$$

belongs to $C_{k,\alpha}(\Omega)$ $(k \geqslant 3,\ \alpha > 0)$, if $F(x, u, p_i)$, as a function of all its arguments, belongs to $C_{k,\alpha}$ and satisfies only the "natural" assumptions formulated in (¹,²).

A survey of works on the questions discussed here is given in (²). The results of the present note, as far as we know, are new even for $n = 2$. Theorem 3 was proved in Morrey' s work (⁶), which has only just appeared, for $n = m = 2$.

Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.