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Abstract

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UNIFORM ESTIMATES OF DERIVATIVES OF SOLUTIONS OF THE DIRICHLET PROBLEM AND OF THE EIGENFUNCTION PROBLEM FOR THE OPERATOR

$$Lu = \operatorname{div}(p(x)\operatorname{grad} u) + q(x) \cdot u$$

WITH DISCONTINUOUS COEFFICIENTS

(Presented by Academician I. G. Petrovskii on 14 X 1960)

In the present note uniform estimates are established for derivatives of arbitrary order and their Hölder coefficients for solutions of the Dirichlet problem and of the eigenfunction problem for the operator
$Lu = \operatorname{div}(p(x)\operatorname{grad} u) + q(x) \cdot u$ with discontinuous coefficients.

1. Let there be given an open N-dimensional domain g with boundary manifold Γ_2, and inside it an $(N-1)$-dimensional geometrically closed surface Γ_1, dividing the domain g into subdomains g_1 and g_2.

Consider in the closed domain $(g + \Gamma_2)$ the following Dirichlet problem:

$$\begin{gathered} L_1 u = f_1 \quad \text{in the domain } g_1, \\ L_2 u = f_2 \quad \text{in the domain } g_2, \\ [u]_{\Gamma_1} = \varphi, \qquad \left[\frac{\partial u}{\partial \nu}\right]_{\Gamma_1} = \psi, \qquad u|_{\Gamma_2} = \chi. \end{gathered} \tag{1}$$

Here

$$L_{lu} = \operatorname{div}(p_l(x)\operatorname{grad} u) + q_l(x) \cdot u \tag{2}$$

is an elliptic operator given in the domain g_l $(l = 1, 2)$, $p_l > 0$, $q_l \leqslant 0$ in g_l;

$$[u]_{\Gamma_1} \equiv u|_{x\to\Gamma_1-0} - u|_{x\to\Gamma_1+0}; \qquad \left[\frac{\partial u}{\partial \nu}\right]_{\Gamma_1} \equiv \left.\frac{\partial u}{\partial \nu_1}\right|_{x\to\Gamma_1-0} + \left.\frac{\partial u}{\partial \nu_2}\right|_{x\to\Gamma_1+0},$$

where $\frac{\partial}{\partial \nu_l}$ denotes differentiation in the direction of the conormal, equal to

$$p_l \frac{\partial}{\partial n_l}$$

(n_l is the outward normal for the domain g_l), and the signs $\Gamma_1 - 0$ and $\Gamma_1 + 0$ mean that the limiting values are taken respectively from the inner and from the outer side of the surface Γ_1 with respect to the domain g_1.

Definition. We shall call a function $u(x)$ a **classical solution** of problem (1) if it satisfies the following requirements: 1) $u(x) \in C^{(0)}$ in each of the closed domains $(g_1 + \Gamma_1)$ and $(g_2 + \Gamma_1 + \Gamma_2)$; 2) $u(x) \in C^{(1)}$ in the domains $(g_1 + \Gamma_1)$ and $(g_2 + \Gamma_1)$; 3) $u(x) \in C^{(2)}$ in the open domains g_1 and g_2; 4) $u(x)$ satisfies all conditions of problem (1) in the usual classical sense.

Theorem 1. *Let the boundary manifolds Γ_1 and Γ_2 belong to the class $A^{(n,\mu)}$, and let the coefficients of the operators L_1 and L_2 and the functions f_1, f_2, φ, ψ and χ satisfy the requirements:*
$p_l \in C^{(n,\mu)}$, $q_l \in C^{(n-2,\mu)}$, $f_l \in C^{(n-2,\mu)}$ *in the closed*

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* The classes $A^{(n,\mu)}$ and $C^{(n,\mu)}$ are defined in (1), p. 10.

smooth domain $(g_1 + \Gamma_1)$; $p_2 \in C^{(n,\mu)}$, $q_2 \in C^{(n-2,\mu)}$, $f_2 \in C^{(n-2,\mu)}$ in the closed domain $(g_2 + \Gamma_1 + \Gamma_2)$; $\varphi \in C^{(n,\mu)}$ and $\psi \in C^{(n-1,\mu)}$ on the surface Γ_1; $\chi \in C^{(n,\mu)}$ on the surface Γ_2 $(n \geqslant 2)$.

Then there exists a unique classical solution of problem (1)—a function $u(x)$; $u(x)$ belongs to the class $C^{(n,\mu)}$ in each of the closed domains $(g_1 + \Gamma_1)$ and $(g_2 + \Gamma_1 + \Gamma_2)$, and in these domains the estimate* holds

$$u^{(n,\mu)} = O\left\{F^{(0)} + \varphi^{(0)} + \psi^{(0)} + \chi^{(0)} + \sum_{i=1}^{n} \psi^{(i-1,\mu)} + \sum_{i=0}^{n} \varphi^{(i,\mu)} + \right.$$
$$\left. + \sum_{i=0}^{n} \chi^{(i,\mu)} + \sum_{i=2}^{n} F^{(i-2,\mu)} \right\}. \tag{3}$$

The constant entering the estimate O depends on the coefficients of the operators L_1 and L_2 and on the form of the domains g_1 and g_2.

For functions belonging to the class $C^{(n,\mu)}$, the following estimate is known (see (1), p. 137)

$$u^{(n)} = O\left\{\left[u^{(n,\mu)}\right]^{\frac{n}{n+\mu}} \left[u^{(0)}\right]^{\frac{\mu}{n+\mu}} + u^{(0)}\right\}. \tag{4}$$

Relying on formulas (3) and (4) and on the estimate of $u^{(0)}$ obtained in (2), we easily estimate $u^{(n)}$.

Remark 1. The estimates $u^{(n,\mu)}$ and $u^{(n)}$ are uniform with respect to operators L_l with uniformly bounded values of the quantities $1/p_l$, $p_l^{(n,\mu)}$, $q_l^{(n-2,\mu)}$, $p_l^{(0)}$, $q_l^{(0)}$ $(l = 1, 2)$.

Remark 2. Theorem 1 is readily extended to the case when the operator L has the form

$$L_{lu} = \operatorname{div}(p_l(x)\operatorname{grad} u) + \sum_{i=1}^{N} b_{li}(x)\frac{\partial u}{\partial x_i} + q_l(x)u.$$

2. The proof of Theorem 1 is carried out as follows. Using the existence theorems proved in (3), we represent the solution of problem (1) in the form of the sum of three functions $u = v_1 + v_2 + w$. The functions v_1 and v_2 are solutions of two ordinary Dirichlet problems, and therefore the known results of Schauder and Caccioppoli (see (1)) may be used for their estimates, while $w(x)$ is the solution of the following problem with discontinuous coefficients:

$$\begin{aligned} &\tilde{L}w = 0 && \text{in the domain } g_1, \\ &\Delta w = 0 && \text{in the domain } g_2, \\ &[w]\big|_{\Gamma_1} = 0, \quad \left[\frac{\partial w}{\partial \nu}\right]\bigg|_{\Gamma_1} = \theta, \quad w\big|_{\Gamma_2} = 0. \end{aligned} \tag{5}$$

Here Δ is the Laplace operator,

$$\tilde{L} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{p_1}{p_2}\frac{\partial}{\partial x_i}\right) + \frac{q_1}{p_2}$$

(the coefficient $p_2(x)$ is extended to the whole closed domain $(g + \Gamma_2)$ with preservation of membership in the class $C^{(n,\mu)}$, which is possible by known results of Gever; see (1), p. 52).

Let $F(x, y)$ denote the Green' s function of the Dirichlet problem for the Laplace operator in the closed domain $(g + \Gamma_2)$. Define the function $\tilde{F}(x, y)$ by means of

* If $z(x)$ is some function defined in a domain T and belonging there to the class $C^{(n,\mu)}$, then by $z^{(k)}$ we shal

by means of the equality

$$\widetilde{F}(x, y) = F(x, y) + \int_{g_1} F(x, t)\,\frac{p_2(t)}{p_1(t)}\,\widetilde{L}_t F(t, y)\,dt. \tag{6}$$

We shall seek the solution of problem (5) in the form of the sum of a volume potential and a single-layer potential

$$w(x) = \int_{g_1} F(x,y)\,\mu(y)\,dy + \int_{\Gamma_1} \widetilde{F}(x,s)\,\nu(s)\,ds \tag{7}$$

with unknown functions μ and ν. Substituting (7) into (5), we arrive at the following system of integral equations:

$$\begin{aligned} \mu(x) - \int_{g_1} K_{11}(x,y)\mu(y)\,dy - \int_{\Gamma_1} K_{12}(x,s)\nu(s)\,ds = 0, \\ \nu(s) - \int_{g_1} K_{21}(s,y)\mu(y)\,dy - \int_{\Gamma_1} K_{22}(s,s_1)\nu(s_1)\,ds_1 = \theta(s). \end{aligned} \tag{8}$$

Formulas (7) and (8) make it possible to estimate the function $w(x)$ by means of the known theorems of potential theory ([4]).

3. Let us consider the eigenfunction problem for the operator $Lu = \operatorname{div}(p(x)\operatorname{grad} u) + q(x)\cdot u$ with discontinuous coefficients

$$\begin{gathered} L_1 u + \lambda u = 0 \quad \text{in the domain } g_1, \\ L_2 u + \lambda u = 0 \quad \text{in the domain } g_2, \\ [u]_{\Gamma_1} = 0, \qquad \left[\frac{\partial u}{\partial \nu}\right]_{\Gamma_1} = 0, \qquad u|_{\Gamma_2} = 0. \end{gathered} \tag{9}$$

Relying on Theorem 1 and on the estimate of the eigenfunctions of problem (9) established in ([3]), we arrive at the following theorem.

Theorem 2. If the surfaces Γ_1 and Γ_2 belong to the class $A^{(n,\mu)}$, and the coefficients of the operators L_1 and L_2 satisfy the conditions: $p_1 \in C^{(n,\mu)}$, $q_1 \in C^{(n-2,\mu)}$ in the closed domain $(g_1 + \Gamma)$; $p_2 \in C^{(n,\mu)}$, $q_2 \in C^{(n-2,\mu)}$ in the closed domain $(g_2 + \Gamma_1 + \Gamma_2)$, then the eigenfunctions of problem (9) belong to the class $C^{(n,\mu)}$ in each of the closed domains $(g_1 + \Gamma_1)$ and $(g_2 + \Gamma_1 + \Gamma_2)$, and for them the following uniform estimates hold in the closed domain $(g + \Gamma_2)$:

$$u_l^{(k)} = O\Big(\lambda_l^{\frac{N}{4}+\frac{k}{2}}\Big), \qquad u_l^{(k,\mu)} = O\Big(\lambda_l^{\frac{N}{4}+\frac{k}{2}+\frac{\mu}{2}}\Big) \tag{10}$$

(l is the number of the eigenfunction).

Remark 3. Theorems 1 and 2 are also valid in the case when, inside the surface Γ_2, there lie m surfaces of discontinuity of the coefficients.

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Note: Figure translations are in progress. See original paper for figures.