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Abstract

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ON THE ABSOLUTE CONVERGENCE OF MULTIPLE FOURIER SERIES

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Thanks to the investigations of S. N. Bernstein ([1]), A. Zygmund ([2]), Hardy ([2]), and to the subsequent generalization and development of their results in the works of S. B. Stechkin ([3–6]), as well as A. A. Konyushkov ([7]), at the present time there are known, in a certain sense, exhaustive criteria for the absolute convergence of Fourier series for functions of one variable. The results on the absolute convergence of multiple Fourier series are of a less complete character. In this connection, the corresponding conditions for absolute convergence usually contain restrictions on the function both with respect to each of its variables separately and with respect to various groups of these variables ([8–17]).

The question naturally arises as to what properties of a function only with respect to each of the variables separately are connected with the absolute convergence of its Fourier series. For functions belonging on the cube of periods to the Lebesgue classes $L_p (1 \leq p \leq \infty)$, such characteristics of their properties may be provided by the partial best approximations of the function $f(x_1, \dots, x_k)$

$$E_{n_\nu,\infty}(f)_{L_p} =$$

$$= \inf_{\varphi_\mu \in L_p} \left\| f(x_1, \dots, x_k) - \sum_{-n_\nu}^{n_\nu} \varphi_\mu(x_1, \dots, x_{\nu-1}, x_{\nu+1}, \dots, x_k) e^{i\mu x_\nu} \right\|_{L_p}$$

or by the partial moduli of smoothness

$$\omega_r(f; h_\nu)_{L_p} = \sup_{|t| \leq h_\nu} \left\| \sum_{\mu=0}^{r} (-1)^{r-\mu} \binom{r}{\mu} f(x_1, \dots, x_{\nu-1}, x_\nu + \mu t, x_{\nu+1}, \dots, x_k) \right\|_{L_p} .$$

It is not difficult to verify that the conditions usually imposed on the function $f(x_1, \dots, x_k)$ with respect to each of its variables separately and ensuring the absolute convergence of the corresponding simple Fourier series are still insufficient for the absolute convergence of the multiple Fourier series of the function $f(x_1, \dots, x_k)$. To this end, consider, for example, the function

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin mx \sin ny}{m^2 + n^2}.$$

It is evident that in this case

$$E_{m,\infty}(f)_{L_2} = O\left(\frac{1}{m}\right),$$

$$E_{\infty,n}(f)_{L_2} = O\left(\frac{1}{n}\right),$$

owing to which the series

$$\sum_{m=1}^{\infty} m^{-1/2} E_{m,\infty}(f)_{L_2}$$

and

$$\sum_{n=1}^{\infty} n^{-1/2} E_{\infty,n}(f)_{L_2}$$

converge, and, consequently ([3]), the Fourier series of the function $f(x, y)$ with respect to the variable x (for almost all y) and with respect to the variable y (for almost all x) converge absolutely. However, this series will not be absolutely convergent.

Thus, the properties of the function $f(x_1, \dots, x_k)$ in each of the variables separately that ensure the absolute convergence of its Fourier series must, in a certain sense, be stronger than the corresponding properties for a function of one variable.

Theorem 1. Let $f(x_1, \dots, x_k) \in L_2$ be periodic with period 2π in each of the variables, and let $c_{n_1,\dots,n_k}$ be its Fourier coefficients. If, for some system of numbers $\alpha_\nu > 0$ $(\nu = 1, 2, \dots, k)$, $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$, and for every $\nu = 1, 2, \dots, k$ the conditions

$$\sum_{n_\nu=1}^{\infty} n_\nu^{-1/2} E^{\alpha_\nu}_{n_\nu,\infty}(f)_{L_2} < \infty, \tag{1}$$

are satisfied, then the series

$$\sum_{-\infty}^{\infty} \cdots \sum_{-\infty}^{\infty} |c_{n_1,\dots,n_k}| \tag{2}$$

converges*. If the numbers $|c_{n_1,\dots,n_k}|$ decrease monotonically in each of the indices, then the divergence of at least one of the series

$$\sum_{n_\nu=1}^{\infty} n_\nu^{-1/2} E_{n_\nu,\infty}(f)_{L_2}$$

implies the divergence of the series (2)**.

In Theorem 1, the "weaker" properties of the function with respect to some variables are compensated by its stronger properties with respect to others. In this connection we present another theorem, of a somewhat different character.

Theorem 2. Let $f(x_1, \dots, x_k) \in L_2$. Then, for convergence of the series (2), it is sufficient that, for every $\nu = 1, 2, \dots, k$, the conditions

$$\sum_{n_\nu=1}^{\infty} n_\nu^{k/2-1} E_{n_\nu,\infty}(f)_{L_2} < \infty. \tag{3}$$

be satisfied.

As the following assertion shows, the conditions (3) in Theorem 2 are final in a certain sense.

Theorem 3. Let $\{\alpha_n\}$ be an arbitrary monotonically decreasing sequence of numbers and let $\alpha_n \to 0$ as $n \to \infty$.

If

$$\sum_{n=1}^{\infty} n^{k/2-1} \alpha_n = \infty,$$

then one can specify a continuous function $f(x_1, \dots, x_k)$, periodic in each of the variables, for which

$$E_{n_\nu,\infty} f = O(\alpha_n) \qquad (\nu = 1, 2, \dots, k;\ n_\nu = n),$$

and the series (2) will diverge.

For $k = 1$ this theorem was proved by S. N. Bernstein ([1], p. 166).

Theorems 1 and 2 show which constructive properties of a function in each of the variables separately ensure the absolute convergence of its Fourier series.

The following two propositions, respectively equivalent to Theorems 1 and 2, express these properties in structural terms***.

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* For $k = 1$, i.e. for a function of one variable, this follows from the inequality (see ([18], p. 303))

$$\sum_{n=1}^{\infty} |c_n| \leq A \sum_{n=1}^{\infty} n^{-1/2} \left(\sum_{n}^{\infty} c_\nu^2 \right)^{1/2} .$$

** For $k = 1$ this assertion was proved by S. B. Stechkin ([4]).

*** For $k = 1$ see ([3]).

Theorem 4. The convergence of the series

$$\sum_{n_\nu=1}^{\infty} n_\nu^{-1/2} \omega_{r_\nu}^{\alpha_\nu} \left(f; \frac{1}{n_\nu} \right)_{L_2} \qquad \left(\nu = 1, 2, \dots, k;\ r_\nu > \frac{1}{2\alpha_\nu} \right) \tag{4}$$

for some $\alpha_\nu > 0$, $\sum_{\nu=1}^{k} \alpha_\nu = 1$, ensures the absolute convergence of the Fourier series of the function $f(x_1, \dots, x_k)$.

Theorem 5. If, for the function $f(x_1, \dots, x_k) \in L_2$, the conditions

$$\sum_{n_\nu=1}^{\infty} n_\nu^{k/2-1} \omega_k \left(f; \frac{1}{n_\nu} \right)_{L_2} < \infty \qquad (\nu = 1, 2, \dots, k), \tag{5}$$

are satisfied, then the series (2) converges.

Since in Theorem 4 only one of the α_ν can be greater than 1/2, while all the remaining α_ν, consequently, are less than 1/2, the absolute convergence of the multiple Fourier series is connected primarily ($r_\nu > 1/2\alpha_\nu$) with the character of the decrease of the moduli of smoothness of the function in each of its variables. This is also confirmed by Theorem 5. At the same time, for functions of one variable it is sufficient to consider only their moduli of continuity.

In conclusion we give some of the possible generalizations of Theorems 1 and 2.

Theorem 6*. Let $f(x_1, \dots, x_k) \in L_2$ be a function periodic with period 2π in each of the variables; let $\Phi(t)$ be an increasing concave function and $\Phi(0) = 0$. Then the inequality

$$\sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \Phi\{|c_{n_1,i_1,\dots,n_k,i_k}|^2\} \leq A \sum_{\nu=1}^{k} \sum_{i_\nu=1}^{\infty} \Phi \left\{ \frac{E_{n_\nu,i_\nu-1,\infty}^{2\alpha_\nu}(f)_{L_2}}{i_\nu} \right\},$$

holds, where $c_{n_1,\dots,n_k}$ are the Fourier coefficients of the function $f(x_1,\dots,x_k)$, $\{n_\nu, i_\nu\}$ ($\nu = 1,2,\dots,k$) are increasing sequences of integers, $\alpha_\nu > 0$ ($\nu = 1,2,\dots,k$), $\sum_{\nu=1}^{k} \alpha_\nu = 1$.

Theorem 7.** Let $f(x_1,\dots,x_k) \in L_p$, $1 < p \le 2$. If, for some system of numbers $\alpha_\nu > 0$ ($\nu = 1,2,\dots,k$), $\sum_{\nu=1}^{k} \alpha_\nu = 1$, for every $\nu = 1,2,\dots,k$ the conditions

$$\sum_{n_\nu=1}^{\infty} n_\nu^{\delta_\nu - \beta(p-1)/p} E_{n_\nu,\infty}^{\beta\alpha_\nu}(f)_{L_p} < \infty,$$

or, equivalently,

$$\sum_{n_\nu=1}^{\infty} n_\nu^{\delta_\nu - \beta(p-1)/p} \omega_{r_\nu}^{\beta\alpha_\nu}\left(f; \frac{1}{n_\nu}\right)_{L_p} < \infty,$$

are satisfied, where

$$r_\nu > \frac{p-1}{p\alpha_\nu}$$

(r_ν are integers), $0 \le \delta_\nu < \beta(p-1)/p$ ($\nu = 1,2,\dots,k$), $0 < \beta < \dfrac{p}{p-1}$, then

$$\sum_{-\infty}^{\infty} \cdots \sum_{-\infty}^{\infty} |c_{n_1,\dots,n_k}|^\beta (|n_1|+1)^{\delta_1} \cdots (|n_k|+1)^{\delta_k} < \infty.$$

* For $k = 1$ see (3).
** For $k = 1$ see (7).

Theorem 8. If $f(x_1,\dots,x_k) \in L_p$, $1 < p \le 2$, and

$$\sum_{n_\nu=1}^{\infty} \omega_k^\beta\left(f; \frac{1}{n_\nu}\right)_{L_p} n_\nu^{k(1-\beta(p-1)/p)-1} < \infty \qquad (\nu = 1,2,\dots,k),$$

where $0 < \beta < \dfrac{p}{p-1}$, then the series

$$\sum_{-\infty}^{\infty} \cdots \sum_{-\infty}^{\infty} |c_{n_1,\dots,n_k}|^\beta$$

converges.

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Note: Figure translations are in progress. See original paper for figures.