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Abstract

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WEAK TOPOLOGY AND PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Solutions of differential equations given in Banach spaces, in some cases, depend weakly continuously on the initial conditions. Comparing this fact with the existence of various fixed-point theorems for weakly continuous mappings (Schauder' s theorem ([1]) and Tikhonov' s ([2]); theorems based on the notion of rotation ([3])), we obtain the possibility of proving the existence of periodic solutions.

1. We make several remarks on the weak topology. Denote by B a real Banach space, and by B^* its conjugate. Denote by N some linear subspace of functionals from B^*. With the aid of N, in the usual way, a weak topology is defined in B, which we shall call the N-weak topology.

Below we assume that every linear bounded functional from N^* can be represented in the form $a(f) = f(x)$, $x \in B$, and that for every x there is such a functional $f \in N$ that $f(x) = \|x\|$, while $\|f\| \leq M$ independently of x.

Then the weak topology satisfies the usual properties: B is a weakly complete space, every weakly convergent sequence is bounded, the unit ball ($\|x\| \leq 1$) is a Hausdorff connected bicompact space in the N-weak topology.

From A. N. Tikhonov' s fixed-point theorem there follows a generalization of Schauder' s theorem ([1]) and its modifications given recently:

Theorem 1. *An operator $F(x)$, acting in the ball $\|x\| \leq 1$ or in a bounded convex weakly closed set and continuous in the N-weak topology, has a fixed point.*

It is known that closures of a set with respect to weak convergence and with respect to the weak topology are, in general, not equivalent. However, if N is separable, then the weak closure of a bounded set $T \subset B$ in the N-weak topology is equivalent to adjoining to T all weak limits of sequences from T.

2. Let $T \subset B$ be some convex weakly closed set. If T is an unbounded set, then we shall assume the existence of such functionals $f_1, \dots, f_p \subset N$ for which the "polyhedron"

$$R = \{x\}, \qquad a_k \leq f_k(x) \leq b_k \quad (k = 1, 2, \dots, p) \tag{1}$$

intersects T in a bounded set $T(R)$ for arbitrary $a_k \leq b_k$, and every bounded set from T is included in one of the R. Introduce in T the N-weak topology. Bounded open sets in this topology will be denoted by U, their boundary by $\dot{U}$, and their closure by $\overline{U}$.

Consider on U a vector field $x - F(x)$, continuous in the N-weak topology, $F(\overline{U}) \subset T$, with no zero vectors on the boundary $\dot{U}$. In addition we shall assume that the operator $F(x)$ is bounded, i.e., maps a bounded set also into a bounded one. Then on the boundary there is defined the rotation

of this field (which is defined in the same way as in (3) for the case of a Hilbert space), which is an invariant of the homotopy of the field, continuous in the weak topology, and which depends additively on the domain U. In the case of a finite number of fixed points $x = F(x)$ in U, the rotation of the field is equal to the sum of their indices. We note that if the rotation is different from zero, then there necessarily exists a fixed point in U.

Let us formulate two theorems on rotation.

Theorem 2. Let U be a set star-shaped with respect to some point $x_0 \in U$, and let $F(x)$ have no fixed points on the boundary $\dot{U}$, with $F(\dot{U}) \subset \overline{U}$.

Then the rotation of the field is equal to $+1$.

Theorem 3. Let T be centrally symmetric, and let U be some centrally symmetric set star-shaped with respect to the point θ. If the field $x - F(x)$ is odd on the boundary $\dot{U}$, then its rotation is also odd.

3. Below we consider the differential equation

$$\frac{dx}{dt} = F(t, x) \tag{2}$$

in the space B. A solution of the equation will be called **weak** if $x(t)$ is strongly continuous in t and $\Delta x / \Delta t \to x'(t)$ in the N-weak topology. We shall assume the operator $F(t, x)$ to be weakly continuous in (t, x) in the N-weak topology.

Theorem 4. Let $F(t, x)$ be defined for $t_1 \leq t \leq t_2$ and $\|x - x_0\| \leq r$, be weakly continuous in the aggregate (t, x), and

$$\sup \|F(t, x)\| = M_0 < \infty, \qquad t_1 \leq t \leq t_2, \quad \|x - x_0\| \leq r.$$

Then there exists a weak solution of equation (2), satisfying the condition $x(t_1) = x_0$ and defined on the interval $t_1 \leq t \leq t_1 + r/MM_0$.

We note that, from the point of view of the Tikhonov topology, continual systems of differential equations were considered in ([4]).

Questions of uniqueness and of nonlocal extendability can be investigated according to the well-known scheme ([5]).

Denote by N_1 some set of functionals from N. Suppose that to each $f \in N_1$ there correspond functions $\psi_f(t)$, $L_f(u)$, $\Phi_f(x,t)$, where $\psi_f(t) \geq 0$ is summable on $[t_1, t_2]$, $L_f(u) \geq 0$ and continuous, $0 \leq u \leq \infty$, and the functional $\Phi_f(x,t)$ is continuous (in norm) with respect to (t, x).

Theorem 5. Let the functional $\Phi_f(x,t)$ satisfy the Lipschitz condition

$$|\Phi_f(x,t) - \Phi_f(y,t)| \leq K_f |f(x-y)|, \qquad K_f > 0, \tag{3}$$

be equal to zero for $x = \theta$; and suppose that for each $x \neq \theta$ some functional $\Phi_f(x,t) > 0$ for all t. Let the condition

$$|f[F(1,x) - F(t,y)]| \leq L_f[\Phi_f(x-y,t)]\psi_f(t), \qquad f \in N_1, \tag{4}$$

hold, where

$$\int_\theta \frac{du}{L_f(u)} = \infty.$$

Then the weak solution of equation (2) with a given initial condition is unique on the interval $[t_1, t_2]$.

Suppose now that $L(u)$, $\Phi(x)$, $\psi(t)$ do not depend on f, and that the nonnegative functional $\Phi(x)$ satisfies the condition

$$\|\Phi(x) - \Phi(y)\| \leq K\|x - y\|, \qquad K > 0, \tag{5}$$

and $\Phi(x) \to \infty$ as $\|x\| \to \infty$, while $\psi(t)$ is bounded and summable on $[t_1, t_2]$.

Theorem 6. Let, at each point $x \in B$, the condition of uniqueness of the weak solution of equation (2) be satisfied, and let the inequality

$$\|F(t,x)\| \leq L[\Phi(x)]\psi(t),$$

where

$$\int^\infty \frac{du}{L(u)} = \infty. \tag{6}$$

Then the weak solution $x(t, x_0)$ with initial condition (t_1, x_0) is defined for all $t \in [t_1, t_2]$, and the operator $x(t, x_0)$ is bounded: from $\|x_0\| \leq \rho$ it follows that $\|x(t, x_0)\| \leq R(\rho)$.

Let us note that Theorems 5 and 6 are also true for the infinite interval $[t_1, +\infty)$.

4. The mapping $x(t, x_0)$ in infinite-dimensional spaces is not completely continuous; however, in a number of cases it is continuous in the weak topology. Namely, the following holds:

Theorem 7. Let N be separable, and let the right-hand side of equation (2) depend weakly continuously on (t, x) for $t \in [t_1, t_2]$, $x \in T$, where T is some weakly closed set in B. Suppose: a) the solution $x(t, x_0)$, $x_0 \in T$, is defined and unique on the interval $[t_1, t_2]$; b) the transformation $x(t, x_0)$ is bounded on T.

Then the operator $x(t, x_0)$ depends weakly continuously on (t, x_0).

This result makes it possible to apply theorems on fixed points of weakly continuous transformations to prove the existence and estimate the number of periodic solutions of equation (2) according to the following scheme.

Consider the set T defined in item 2. Suppose that the right-hand side of equation (2) is periodic in t: $F(t+\omega, x) = F(t, x)$, $x \in T$, and that the solutions satisfy the conditions of Theorem 7. Consider the field $x_0 - x(\omega, x_0)$. If there exists a fixed point of this field, $x_0^* = x(\omega, x_0^*)$, then it corresponds to a periodic solution $x(t, x_0^*)$.

Lemma. If the field $x_0 - x(\omega, x_0)$ on the boundary $\dot{U}$ of some bounded open (in the weak topology) set $U \subset T$ has nonzero rotation, then there exists a periodic solution of period ω, beginning in U.

The considerations on which this lemma is based go back to A. Poincaré (see, for example, ([6])). For finite-dimensional cases, a number of works are based on similar arguments (let us note, in particular, the works ([7,8])). The use of the weak topology is apparently indicated here for the first time.

We shall give two theorems on periodic solutions. We shall assume N to be separable and shall first consider the case when the set T is bounded and is a "polyhedron" :

$$T = \{x\}, \quad a_\varphi \le \varphi(x) \le b_\varphi, \quad \varphi \in N_2,$$

where $a_\varphi \le b_\varphi$ are certain numbers, and N_2 is some subset of N.

Theorem 8. Let $F(t, x)$ be an operator weakly continuous in (t, x) in a neighborhood S of the set T, periodic of period ω, satisfying the inequalities:

$$\varphi[F(t, x)] \ge 0, \quad \text{if } \varphi(x) = a_\varphi,\ x \in T;$$

$$\varphi[F(t, x)] \le 0, \quad \text{if } \varphi(x) = b_\varphi,\ x \in T.$$

Suppose that at the points of the set S the condition of uniqueness (4) of the weak solution of equation (2) is satisfied.

Then on the set T there exists a periodic weak solution of period ω.

Now we shall no longer assume T to be bounded, but shall admit the existence of a "polyhedron" (1); moreover we shall confine ourselves to the case $a_\varphi = -b_\varphi$, $\varphi \in N_2$. We shall assume the existence of functionals $\Phi_k(x) \geqslant 0$ $(k = 1, 2, \dots, p)$, satisfying the Lipschitz condition (3), the condition

$$\lim[\Phi_1(x) + \dots + \Phi_p(x)] = +\infty$$

as $x \in T$, and the condition

$$|f_k[F(t,x)]| \leqslant L_k[\Phi_k(x)]\psi_k(t) \qquad (k = 1, 2, \dots, p),$$

where $\psi_k(t) \geqslant 0$ are summable on $[0, \omega]$, while $L_k(u) \geqslant 0$ are continuous and satisfy condition (6). We shall also assume that the conditions of Theorem 8 are fulfilled (except for the boundedness of T).

Let $\Phi(x)$ be a weakly continuous and continuously Fréchet differentiable functional, increasing to $+\infty$ as $\|x\| \to \infty$, $x \in T$, equal to zero only for $x = \theta$, even with respect to x, and $(\operatorname{grad}\Phi(x), x) > 0$ for $\|x\| > 0$, $x \in T$.

Theorem 9. *If the inequality* $(\operatorname{grad}\Phi(x), F(t,x)) \geqslant 0$ *holds whenever* $\Phi(x) = r_0 > 0$, $x \in T$, *and the operator* $F(t,x)$ *is odd in* x *in the domain* $\Phi(x) \geqslant r_0$, *then there exists a periodic weak solution of period* ω.

5. Let us note that the scheme set forth by us covers the case of Orlicz spaces and is apparently valid for locally compact linear topological spaces. In particular, the theorems proved by us can be formulated for infinite systems of differential equations. If the operator $F(t,x)$ is norm-continuous, then a weak solution is necessarily a strong solution. The use of the concept of the rotation of a field makes it possible in some cases to prove the existence of second periodic solutions*.

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* Preliminary results were reported by us at the conference on functional analysis in Baku in 1959.

Note: Figure translations are in progress. See original paper for figures.