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Abstract

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ON THE EXTENSION OF HOMEOMORPHISMS

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In this note, from the theorem on the extension of homeomorphisms proved in ([3,5]), there is derived a result, due to K. Kuratowski and S. Eilenberg, on the extension of homeomorphisms to acyclic extensions*. At the same time one obtains a theorem concerning a broader class of spaces and their bicompact extensions.

Theorem 1. *Let X_1 and X_2 be connected completely regular spaces; let Y_1 and Y_2 be such bicompact extensions of them that the remainders $Y_1 \setminus X_1$ and $Y_2 \setminus X_2$ are punctiform,* ***and*** $H^1(Y_1, Z) = 0$, $H^1(Y_2, Z) = 0$; then every homeomorphism between the spaces X_1 and X_2 extends to a homeomorphism between Y_1 and Y_2.*

In other words, a connected space can have at most one bicompact extension, acyclic in dimension 1, with punctiform remainder.

Proof. We use the following theorem (see ([5])):

If in the bicompact extensions Y_1 and Y_2 the remainders are punctiform and at no point of theirs separate Y_1 and Y_2, then every homeomorphism between the spaces X_1 and X_2 extends to a homeomorphism between Y_1 and Y_2.

By virtue of this theorem, it is enough to show that the remainders $Y_1 \setminus X_1$ and $Y_2 \setminus X_2$ at no point of theirs separate the bicompacts Y_1 and Y_2. In other words, we have to show that an acyclic in dimension 1 bicompact extension Y of a connected space X with punctiform remainder $Y \setminus X$ is indecomposable (in this case $Y = \mu X$, see Theorem 3 of ([5])).

Suppose that $Y \setminus X$ separates Y at some point $y \in Y \setminus X$, i.e. the point y has a neighborhood W such that $W \cap X = W' \cup W''$, where W' and W'' are open in X, $W' \cap W'' = \Lambda$, and $y \in Y[W'] \cap Y[W'']$. Let V be such a neighborhood of the point y that $Y[V] \subset W$. Then the set $N = Y[V] \cap Y[W'] \cap Y[W''] \subset Y \setminus X$ is closed in Y, zero-dimensional (since it is bicompact and punctiform), separates Y at the point y, and the space $X' = Y \setminus N$ is connected.

Choose a neighborhood $U \subset V$ of the point y, whose boundary does not intersect N. Let $U' = (U \cap Y[W']) \setminus N$; U'' is defined analogously. Then $U \setminus N = U' \cup U''$, $U' \cap U'' = \Lambda$, and $y \in Y[U'] \cap Y[U'']$. Let $A = N \cap U$, $B = Y \setminus U$; the sets

A and B are closed in Y and do not intersect. There exists a mapping f of the bicompact Y into the circle S^1 such that the sets A and B are mapped to diametrically opposite points of the circle, and the sets U' and U'' are mapped into the opposite semicircles joining these points. Let u be a one-dimensional fundamental class

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* This result was communicated by K. Kuratowski in his report at the Fourth All-Union Mathematical Congress (Leningrad, July 1961) and by S. Eilenberg at the International Topological Symposium (Prague, September 1961).

** A space is called punctiform if every connected bicompact subset of it consists of a single point.

*** Spectral cohomology is meant.

cohomology of the circle; we shall show that $f^*u \neq 0$, contrary to the assumption that Y is acyclic.

One may assume that f is a barycentric map to the nerve of some covering α. Let β be an arbitrary covering inscribed in α. It suffices to show that $\pi^\alpha_\beta u \neq 0$, where π^α_β is the cohomology mapping corresponding to the projection of the nerve of the covering β to the nerve of the covering α (i.e. to S^1). Choose a neighborhood V_0 of the point y, contained in U and in one of the elements of the covering β. Let z be an arbitrary point in B. Since X' is connected, in Y one can find a chain of open sets $V_1, \dots, V_m$ such that: 1) $V_i \cap N = \Lambda$, $i = 1, \dots, m$; 2) each V_i is contained in some element of the covering β; 3) $V_0 \cap V_1 \neq \Lambda$, $V_{i-1} \cap V_i \neq \Lambda$, $i = 2, \dots, m$, $z \in V_m$; 4) $V_1 \subset U'$; 5) if some $V_i \subset U$, then either $V_i \subset U'$, or $V_i \subset U''$. There also exists a chain $W_1, \dots, W_n$ satisfying the preceding conditions with U' in condition 4) replaced by U''. Let γ be the covering obtained by adjoining to β the sets $V_0, V_1, \dots, V_m, W_1, \dots, W_n$. In the nerve of the covering γ we have a cycle which projects to S^1 with degree 1. Therefore $\pi^\alpha_\gamma u \neq 0$, and since the covering γ is inscribed in β, still more so $\pi^\alpha_\beta u \neq 0$. The theorem is proved.

We shall show that in fact the requirement that the extensions be acyclic in the theorem on extension of homeomorphisms can be replaced by a weaker condition.

Theorem 2. *Let X_1 and X_2 be connected completely regular spaces; let Y_1 and Y_2 be their bicompact extensions with punctiform remainders, and suppose that the natural mappings*

$$H^1(Y_i, \mathbf{Z}) \to \check{H}^1(X_i, \mathbf{Z}), \quad i = 1, 2,$$

are monomorphisms (or, equivalently, the mappings of homology groups with bicompact coefficients are epimorphisms); then every homeomorphism between the spaces X_1 and X_2 extends to a homeomorphism between Y_1 and Y_2. *

Proof. It is enough to show that *if, for a bicompact extension Y of a connected space X with punctiform remainder, $Y \setminus X$, the mapping*

$$\pi^* : H^1(Y, \mathbf{Z}) \to H^1(\beta X, \mathbf{Z})$$

is monomorphic, then the extension Y is perfect (see the footnote).

We shall give two proofs of this fact.

First proof is obtained by a small addition to the preceding argument. Assuming that the extension Y is not perfect, we construct, as above, a mapping $f : Y \to S^1$, determining a nonzero cohomology class $f^*u \in H^1(Y, \mathbf{Z})$. We shall show that $\pi^* f^* u = 0$, contrary to the assumption that the mapping π^* is monomorphic. To this end we prove that the mapping $f \circ \pi : \beta X \to S^1$ is homotopic to a constant. Let

$$F = B \cap X, \quad V' = U' \cap X, \quad V'' = U'' \cap X.$$

The set F is closed in X and separates X into the sets V' and V''. Since the extension βX is perfect, $\beta X[F]$ separates βX into $O\langle V' \rangle$ and $O\langle V'' \rangle$ (see (5), Theorem 1). The mapping $f \circ \pi$ sends the set $\beta X[F]$ to a point, and the sets $O\langle V' \rangle$ and $O\langle V'' \rangle$ to semicircles. Therefore the mapping $f \circ \pi$ is contractible to a point.

Second proof ** does not depend on the preceding arguments

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* For a non-bicompact space X we consider the cohomology groups $\check{H}^*(X, \mathbf{Z})$, defined by means of finite normal coverings, in other words, the groups canonically isomorphic to the cohomology groups of the Čech extension βX. Note that the requirement that the mapping $H^1(Y, \mathbf{Z}) \to \check{H}^1(X, \mathbf{Z})$ be monomorphic is weaker than the requirement that the mapping $H^1(Y, \mathbf{Z}) \to H^1(X, \mathbf{Z})$ be monomorphic, since the latter is represented as the composition

$$H^1(Y, \mathbf{Z}) \to \check{H}^1(X, \mathbf{Z}) \to H^1(X, \mathbf{Z})$$

($\check{H}^*(X, \mathbf{Z})$ denotes cohomology defined by means of all coverings). The mapping

$$i^* : H^1(Y, \mathbf{Z}) \to \check{H}^1(X, \mathbf{Z})$$

is identical with the mapping

$$\pi^* : H^1(Y, \mathbf{Z}) \to H^1(\beta X, \mathbf{Z}),$$

induced by the natural projection $\pi : \beta X \to Y$. Therefore, in what follows, $H^1(\beta X, \mathbf{Z})$ and π^* are used instead of $\check{H}^1(X, \mathbf{Z})$ and i^*.

** As is clear from the arguments given below, the requirement that X be connected can be replaced by a weaker condition; it is enough to assume that the mapping

$$i^* : H^0(Y, \mathbf{Z}) \to \check{H}^0(X, \mathbf{Z})$$

is an epimorphism onto.

arguments, but uses elements of sheaf theory. By virtue of Theorem 2 from ([5]), it suffices to show that the mapping π is monotone. Let Z be the constant sheaf isomorphic to the group of integers (which we shall consider on the bicompacta βX and Y). Let $R^0\pi(Z)$ be the direct image of the sheaf Z under the mapping π (see, for example, ([4])). The stalk of the sheaf $R^0\pi(Z)$ over a point $y \in Y$ is the group $H^0(\pi^{-1}y, Z)$. The mapping π is monotone if and only if, for every point y, $H^0(\pi^{-1}y, Z) = Z$, i.e. if the canonical embedding $Z \to R^0\pi(Z)$ is an isomorphism of sheaves. Consider the exact cohomology sequence

$$0 \to H^0(Y, Z) \to H^0(Y, R^0\pi(Z)) \to H^0(Y, R^0\pi(Z)/Z)$$
$$\to H^1(Y, Z) \to H^1(Y, R^0\pi(Z)) \to \dots, \qquad (*)$$

corresponding to the exact sequence of sheaves

$$0 \to Z \to R^0\pi(Z) \to R^0\pi(Z)/Z \to 0.$$

Since the spaces under consideration are connected, the mapping

$$H^0(Y, Z) \to H^0(Y, R^0\pi(Z)) = H^0(\beta X, Z)$$

is an isomorphism onto. On the other hand, consider the mappings

$$H^1(Y, Z) \to H^1(Y, R^0\pi(Z)) \to H^1(\beta X, Z).$$

Since, by hypothesis, the composite mapping is a monomorphism, all the more so the mapping $H^1(Y, Z) \to H^1(Y, R^0\pi(Z))$ is a monomorphism. Therefore, from the exactness of the sequence $(*)$ we conclude that

$$H^0(Y, R^0\pi(Z)/Z) = 0.$$

We shall show that then $R^0\pi(Z)/Z = 0$. Since the mapping π is the identity at the points of the space X, the sheaf $R^0\pi(Z)/Z$ is concentrated on the remainder $Y \setminus X$. Suppose that at some point $y \in Y \setminus X$

$$\left(R^0\pi(Z)/Z\right)_y \neq 0.$$

Let s be a section of the sheaf $R^0\pi(Z)/Z$, defined over some neighborhood U of the point y, and such that $s_y \neq 0$. Since the sheaf $R^0\pi(Z)/Z$ vanishes at the points of the space X, the set N of those points at which the section s is different from zero is closed in U and is contained in the remainder. Since the remainder is punctiform, $\operatorname{ind} N = 0$; consequently, at the point y there exists a neighborhood $V \subset U$ whose boundary does not meet the set N. Define a global section $\bar{s}$ of the sheaf $R^0\pi(Z)/Z$ by putting $\bar{s}_x = s_x$ if $x \in V$, and $\bar{s}_x = 0$ if $x \in Y \setminus V$. The section $\bar{s} \neq 0$, contrary to the fact that

$$H^0(Y, R^0\pi(Z)/Z) = 0.$$

Thus it has been proved that the extension Y is perfect.

Remark. We have derived Theorem 2 from the theorem on extension of homeomorphisms cited above from ([5]). In fact, for connected spaces these two theorems are equivalent, as the following proposition shows.

If a bicompact extension Y of a completely regular space X is perfect, then the mapping

$$i^* : H^1(Y, Z) \to \check{H}^1(X, Z)$$

is a monomorphism.

Indeed, the mapping $\pi : \beta X \to Y$ is monotone (see ([5]), Theorem 2), i.e. the inverse images of points are acyclic in dimension 0. Therefore our assertion follows from the Vietoris–Begle theorem ([1], [2]).

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Note: Figure translations are in progress. See original paper for figures.