Mathematics

1961

SovietRxiv

Abstract

Full Text

Mathematics

K. K. GOLOVKIN

SOME CONDITIONS FOR SMOOTHNESS OF FUNCTIONS OF SEVERAL VARIABLES AND ESTIMATES OF CONVOLUTION OPERATORS

(Presented by Academician V. I. Smirnov, 13 III 1961)

The results of the present work are based on the following elementary algebraic identities:

$$\frac{1}{2^l}(x^2-1)^l = (x-1)^l + \frac{1}{2^l}(x-1)^{l+1}\sum_{j=1}^{l}\left(\sum_{k=j}^{l} C_l^k\right)x^{j-1}, \quad (1)$$

$$\frac{1}{\sum_{i=1}^m l_i}\prod_{i=1}^m (x_i^n-1)^{l_i} = \prod_{i=1}^m (x_i-1)^{l_i} + \sum_{j=1}^m P_j(x_1,\dots,x_m)\prod_{i=1}^m (x_i-1)^{l_i+\delta_i^j}, \quad (2)$$

where δ_i^j is the Kronecker symbol; P_j are certain polynomials in $x_1,\dots,x_m$, the form of which is immaterial for what follows;

$$\prod_{i=1}^m (x_i-1)^{l_i} = -\prod_{i=1}^m (x_i^n-1)^{l_i}\left(\sum_{s=1}^N a_s \prod_{i=1}^m \frac{x_i^{nl_i}(x_i^{ns}-1)^{l_i}}{(x_i-1)^{l_i}}\right) + \sum_{s=1}^N a_s \prod_{i=1}^m (x_i^{ns+1}-1)^{l_i}. \quad (3)$$

Here n is an arbitrary natural number, $N=\prod_{i=1}^m (l_i+1)-1$, and the functions a_s are polynomials in the negative powers x_i^{-n}. We give the form of these polynomials. To this end, number in an arbitrary order by the numbers from 1 to $N-1$ all possible products

$$x_1^{j_1 n} x_2^{j_2 n} \cdots x_m^{j_m n}$$

for

$$0 \le j_1 \le l_1, \dots, \quad 0 \le j_m \le l_m \quad \text{and} \quad 0 < \sum_{k=1}^m j_k < \sum_{k=1}^m l_k.$$

Denote such a product with the i-th number by ν_i. Further, denote by $Q_j(y_1, \dots, y_{N-1})$ the homogeneous polynomial of order j, symmetric in all arguments and linear in each of them, normalized by the condition

$$Q_j(1, \dots, 1) = C_{N-1}^j.$$

We also put, for uniformity of notation,

$$Q_{-1}(y_1, \dots, y_{N-1}) \equiv 0.$$

Then

$$a_s = (-1)^{s+1} \frac{Q_{N-s}(\nu_1, \dots, \nu_{N-1}) + Q_{N-s-1}(\nu_1, \dots, \nu_{N-1})}{\prod_{j=1}^{N-1} \nu_j}.$$

In the identities (1)–(3), instead of the letters x_i one may substitute shift operators in the i-th variable with arbitrary step h. Then the binomial $(x_i - 1)^{l_i}$ becomes the operator of taking the finite difference $\Delta_{h,i}^{l_i}$. As a result one obtains ...

certain operator identities which, in the cases (1), (2), relate finite differences of a definite kind with step h to differences of the same kind with step increased by a factor of 2 or by a factor of n, and to differences of higher order; while in the case (3) one obtains an expression of a mixed difference with step h in terms of differences of the same kind but with arbitrarily large steps. We shall assume throughout the sequel that the functional norm $\| \ \|$, defined for functions of m variables, is invariant with respect to translation operators in each variable. Then the operator identities (1)–(3) give us, respectively, the inequalities:

$$\|\Delta_{h,i}^{l} u\| \le \frac{1}{2^l} \|\Delta_{2h,i}^{l} u\| + C \|\Delta_{h,i}^{l+1} u\|; \tag{4}$$

$$\|\Delta_{h,1}^{l_1} \cdots \Delta_{h,m}^{l_m} u\| \le$$

$$\le \frac{1}{n^{\sum_{k=1}{}^{m} l_k}} \|\Delta_{nh,1}^{l_1} \cdots \Delta_{nh,m}^{l_m} u\| + C \sum_{i=1}^{m} \|\Delta_{h,1}^{l_1} \cdots \Delta_{h,i}^{l_i+1} \cdots \Delta_{h,m}^{l_m} u\|; \tag{5}$$

$$\|\Delta_{h,1}^{l_1} \cdots \Delta_{h,m}^{l_m} u\| \le$$

$$\le C \left[\sum_{s=1}^{N} \|\Delta_{(ns+1)h,1}^{l_1} \cdots \Delta_{(ns+1)h,m}^{l_m} u\| + \|\Delta_{nh,1}^{l_1} \cdots \Delta_{nh,m}^{l_m} u\| \right], \tag{6}$$

It is important to note that in inequality (5) the constant depends on l_i and on n, while in inequality (6) it depends only on l_i. Inequality (4) is a simple and important special case of (5).

We shall call a norm J, defined for nonnegative piecewise-continuous functions $\psi(h)$ given on the half-line $h > 0$, vanishing in some neighborhood of the point $h = 0$, and decreasing as a negative power of h as $h \to \infty$, a functional of maximization type if it has the following properties:

1. From the inequality $\psi(h) \geq \varphi(h)$ it follows that $J[\psi(h)] \geq J[\varphi(h)]$.
2. $J[\psi(kh)] = J[\psi(h)]$.

On the basis of property 1 one may extend the domain of definition of a functional of maximization type, including in it some functions which do not vanish in a neighborhood of zero. Namely, put

$$J[\psi(h)] = \lim_{\xi \to 0} J\left[\varepsilon\left(\frac{h}{\xi}\right)\psi(h)\right],$$

where

$$\varepsilon(y) = \begin{cases} 1, & y > 1, \\ 0, & y \leq 1, \end{cases}$$

if the written limit is finite. It is easy to see that, after such an extension of the domain of definition, J still possesses properties 1 and 2. The most important functionals of maximization type are

$$J_p[\psi(h)] = \left\{\int_0^\infty \frac{\psi^p(h)}{h}\,dh\right\}^{1/p}, \quad p \geq 1; \qquad J_\infty[\psi(h)] = \sup_{h>0} \psi(h).$$

Bearing in mind that the operator $\partial^l/\partial x_i^l$ admits in the norm $\| \; \|$ the representation

$$\lim_{h \to 0} \frac{\Delta_{h,i}^l}{h^l},$$

which is easily justified using its translation invariance, one can derive from inequality (4) the results stated below. Suppose that r is any positive number, r' is the greatest integer less than r, and r'' is the least integer greater than r.

Theorem 1. In the class of functions for which*

$$J\left[\frac{\varepsilon(h)}{h^2}\left\|\Delta_{g,\bar{h}}^{\,r} u\right\|\right] < \infty,$$

all norms of the form

$$J\left[\frac{1}{h^{r-s}}\left\|\Delta_{h,i}^{l}\frac{\partial^s u}{\partial x_i^s}\right\|\right],$$

where l, s are integers such that $l > r - s > 0,\ s \geq 0$, are equivalent to one another.

On the basis of inequalities (5) and (6), more general theorems can be proved.

Theorem 2. Let $l_1, \dots, l_m$ be an arbitrary set of natural numbers and let $l_1 + \dots + l_{m'} > r$ for some $m' \leq m$. Then, in the class of functions for which

$$J\left[\frac{\varepsilon(h)}{h^r}\left\|\Delta_{h,1}^{l_1}\cdots\Delta_{h,m}^{l_m}u\right\|\right] < \infty,$$

all norms of the form

$$\sum_{j=1}^{m'} J\left[\frac{1}{h^r}\left\|\Delta_{h,1}^{l_1}\cdots\Delta_{h,j}^{l_j+k}\cdots\Delta_{h,m}^{l_m}u\right\|\right],$$

where k is any integer and nonnegative number, are equivalent.

Theorem 3. Let $0 < r \leq 1$; let $l_1, \dots, l_{m'}$ be arbitrary natural numbers whose sum is l, with $m' \leq m$. Then, in the class of functions for which

$$\sum_{j=1}^{m'} J\left[\frac{\varepsilon(h)}{h^{r+l}}\left\|\Delta_{h,1}^{l_1}\cdots\Delta_{h,j}^{l_j+2}\cdots\Delta_{h,m'}^{l_{m'}}u\right\|\right] < \infty$$

all norms of the form

$$\sum_{j=1}^{m'} J\left[\frac{1}{h^{r+l}}\left\|\Delta_{h,1}^{l_1}\cdots\Delta_{h,j}^{l_j+k}\cdots\Delta_{h,m'}^{l_{m'}}u\right\|\right]$$

are equivalent to the norm

$$\sum_{j=1}^{m'} J\left[\frac{1}{h^r}\left\|\Delta_{h,j}^{2}\frac{\partial^l u}{\partial x_1^{l_1}\cdots\partial x_{m'}^{l_{m'}}}\right\|\right].$$

From a comparison of Theorems 1 and 3 the following result follows:

Theorem 4. Under the conditions of Theorem 3, the estimate

$$\sum_{i=1}^{m'} J\left[\frac{1}{h^r}\left\|\Delta_{h,i}^{2}\frac{\partial^l u}{\partial x_1^{l_1}\cdots\partial x_{m'}^{l_{m'}}}\right\|\right] \leq C\sum_{i=1}^{m'} J\left[\frac{1}{h^r}\left\|\Delta_{h,i}^{2}\frac{\partial^l u}{\partial x_i^l}\right\|\right]$$

holds.

In particular, if $\| \ \|$ is understood as the norm in L_p over the whole space or over a hyperplane and one sets $J = J_\infty$, then the assertion of the last theorem coincides with that special case of Theorem 13 in the work of S. M. Nikol' skii [1] in which the differential properties of the function are the same in all directions. If, however, one sets $\| \ \| = \| \ \|_{L_p}$ and $J = J_p$, then we arrive at an analogous special case of the theorem recently established by V. I. Il' in and V. A. Solonnikov for the spaces W_p^r.

We pass to estimates of convolution operators. Here we shall restrict ourselves to the formulation of the simplest results following from Theorem 1, and shall not touch upon more subtle ones connected with the application of Theorems 2 and 3. Let E_m be the space of m dimensions and

$$u(x) = \int_{E_m} G(x-y)\varphi(y)\,dy.$$

* In essence this assumption concerns the behavior of the function at infinity.

Theorem 5. Suppose that for some $l > \alpha > 0$

$$\int_{E_m} |\Delta_{h,i}^{l} G(y)|\,dy < Ch^\alpha$$

and suppose that for some $r > \alpha$

$$J\left[\frac{\varepsilon(h)}{h^r}\int |\Delta_{h,i}^{r''} G(y)|\,dy\right] < \infty.$$

Then, if $\|\varphi(y)\| < \infty$, then

$$J\left[\frac{1}{h^{r'-r}}\left\|\Delta_{h,i}^{r''-r'}\frac{\partial^{r'}u}{\partial x_i^{r'}}\right\|\right] \le C_1 J\left[\frac{1}{h^{r'-\alpha}}\|\Delta_{h,i}^{k}\varphi\|\right],$$

where $k > r - \alpha$.

Theorem 6. Suppose that for some $l > \alpha$

$$\int |\Delta_{h,i}^{l} G(y)|\,dy \le Ch^\alpha$$

and k is an integer not less than l. Then, if $\|\varphi\| > \infty$, then

$$\left\|\frac{\partial^k u}{\partial x_i^k}\right\| \le C_1 J_1\left[\frac{1}{h^{k-\alpha}}\|\Delta_{h,i}^{s}\varphi\|\right],$$

where $s > k - \alpha$.

Example. Let $G(y) = 1/r(y)^p$ $(p < m)$. This kernel differs only by a constant factor from the kernel of the operator $(-\Delta)^{-(m-p)/2}$. The conditions of Theorem 5 are satisfied with $\alpha = m - p$ and with any $r > \alpha$. Put, for example, $J = J_\infty$ and $\| \: \| = \| \: \|_{L_p}$. By Theorem 5 we obtain

$$\|u\|_{H_q^{(r_1 \dots r_m)}} \le C\|\varphi\|_{H_q^{(r_1-\alpha \dots r_m-\alpha)}}.$$

From Theorem 6 in this case it follows that the domain of definition in L_q of the operator $(-\Delta)^{\beta/2}$ contains all functions with finite norm

$$\int_0^\infty \frac{1}{h^{\beta+1}} \|\Delta_{h,i}^s \varphi\|_{L_q} \, dh, \qquad s > \beta.$$

For $q = 1$ this result is final, while for $q > 1$ it is, in any case, sharp with respect to the differential order and to the dimension.

Application of Theorem 5 in the one-dimensional case for $G(y) = 1/y_+^\lambda$ makes it possible to extend Theorem 1 also to fractional orders of differentiation s. The results thereby obtained contain, as special cases, the results of Hardy–Littlewood ([2]) and Zygmund ([3]) on fractional integration.

Theorems 2-4 generalize to strongly continuous m-parameter semigroups of uniformly bounded operators acting in normed spaces.

In conclusion, we note that some special cases of the theorems stated here were established by A. Zygmund ([3]), S. N. Bernstein ([4]), A. F. Timan and M. F. Timan ([5]). All the named authors relied on the apparatus of the theory of best approximations, which is not used at all in the present work. Instead, certain combinatorial identities are used, which are most easily discovered by means of elementary algebra.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
28 II 1961

References

1. S. M. Nikol' skii, *Trudy Mat. Inst. im. V. A. Steklova AN SSSR*, **38** (1951).

2. G. H. Hardy, S. E. Littlewood, *Math. Zs.*, **27**, 565 (1928).

3. A. Zygmund, *Duke Math. J.*, **12**, 47 (1928).

4. S. N. Bernstein, *DAN*, **59**, No. 8 (1948).

5. A. F. Timan, M. F. Timan, *DAN*, **113**, No. 5 (1957).

Note: Figure translations are in progress. See original paper for figures.