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Abstract

Full Text

MATHEMATICS

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**ON PROJECTION IN A HILBERT SPACE
WITH A BILINEAR METRIC**

(Presented by Academician L. S. Pontryagin on 21 III 1961)

Let, in a Hilbert space \mathfrak{H} with scalar product (f, g) , a continuous bilinear (generally speaking, not Hermitian-symmetric) functional be given,

$$[f, g] = (Gf, g).$$

We shall call the bounded linear operator G the **Gram operator** of the functional $[f, g]$ in the space \mathfrak{H} .

Let \mathfrak{M} be a linear manifold in \mathfrak{H} . A vector $g \in \mathfrak{M}$ will be called a (left) G -projection of the vector f onto \mathfrak{M} if $[f - g, h] = 0$ for all $h \in \mathfrak{M}$. An operator $Q_{\mathfrak{M}}$ (generally speaking, multivalued) will be called a G -projector onto \mathfrak{M} if to each $f \in \mathfrak{H}$ it assigns the set (possibly empty) of G -projections of f onto \mathfrak{M} . Some sufficient conditions for the existence of G -projections in the case of Hermitian-symmetric $[f, g]$ were obtained by R. Nevanlinna ⁽¹⁾ and I. S. Louhivaara ⁽²⁾. In the case where the form $[f, f]$ has a finite number of negative squares, this question was considered earlier by L. S. Pontryagin ⁽³⁾, and then by I. S. Iokhvidov and M. G. Krein ^(4,5).

In the present note a formula is given which relates the G -projector onto a linear manifold \mathfrak{M} to an operator T whose range coincides with \mathfrak{M} , and a number of criteria for the existence and uniqueness of G -projections are formulated, from which, in particular, the aforementioned results of Nevanlinna and Louhivaara follow. In addition, for the special case $G = J$ ($J^2 = I$, $J^* = J$) we give a complete description of those definite (positive and negative ⁽⁴⁾) subspaces \mathfrak{M} for which the operator $Q_{\mathfrak{M}}$ is defined on all of \mathfrak{H} .

1. Let A be an arbitrary linear operator in \mathfrak{H} . By A^{-1} we shall henceforth mean the operator (generally speaking, multivalued) which assigns to each $f \in \mathfrak{H}$ the set (possibly empty) of all its preimages h ($Ah = f$).

Theorem 1. If \mathfrak{M} is the range of a bounded linear operator T , then

$$Q_{\mathfrak{M}} = T (T^*GT)^{-1}T^*G.$$

If \mathfrak{M} is a subspace, then, putting $T = P_{\mathfrak{M}}$, where $P_{\mathfrak{M}}$ is the usual orthogonal projector onto \mathfrak{M} , we obtain

$$Q_{\mathfrak{M}} = G_{\mathfrak{M}}^{-1} P_{\mathfrak{M}} G$$

(here $G_{\mathfrak{M}} = P_{\mathfrak{M}} G P_{\mathfrak{M}}$ is the Gram operator of the functional $[f, g]$ in the subspace \mathfrak{M}).

Hence the following assertions follow directly:

1°. In order that each vector of \mathfrak{H} have no more than one G -projection onto the subspace \mathfrak{M} , it is necessary and sufficient that on \mathfrak{M}

the operator $G_{\mathfrak{M}}$ not vanish (such a subspace \mathfrak{M} will be called **nondegenerate**).

2°. In order that the vector f_0 have a G -projection onto the subspace \mathfrak{M} , it is necessary and sufficient that

$$P_{\mathfrak{M}} G f_0 \in \mathfrak{R}(G_{\mathfrak{M}})$$

(by the symbol $\mathfrak{R}(A)$ we denote the range of the operator A).

3°. Every vector in \mathfrak{H} has at least one G -projection onto the subspace \mathfrak{M} if and only if

$$\mathfrak{R}(P_{\mathfrak{M}} G) = \mathfrak{R}(G_{\mathfrak{M}}).$$

Below we formulate assertions 1° and 2° in another form. To this end, in the subspace $\mathfrak{M} \subseteq \mathfrak{H}$ consider the seminorm ⁽⁶⁾

$$|f|_{\mathfrak{M}} = \|G_{\mathfrak{M}} f\| = (G_{\mathfrak{M}} G_{\mathfrak{M}}^* f, f)^{1/2}$$

(in particular, $|f|_{\mathfrak{H}} = \|G^* f\|$). We shall call the subspace \mathfrak{M} **relatively regular** if the seminorms $|f|_{\mathfrak{M}}$ and $|f|_{\mathfrak{H}}$ are equivalent on \mathfrak{M} , which in the case under consideration reduces to the following: if $|f_n|_{\mathfrak{M}} \rightarrow 0$ ($f_n \in \mathfrak{M}$), then $|f_n|_{\mathfrak{H}} \rightarrow 0$. We shall call the subspace \mathfrak{M} **regular** if on \mathfrak{M} the quantities $|f_n|_{\mathfrak{M}}$ and $\|f\|$ are equivalent. Obviously, every regular subspace is relatively regular. The converse is true only when \mathfrak{H} itself is regular. On the basis of Banach' s theorem on the inverse operator, the subspace \mathfrak{M} is regular if and only if $|f|_{\mathfrak{M}}$ is a norm and in this norm \mathfrak{M} is complete, which for a definite \mathfrak{M} is equivalent to completeness in the norm $[[f, f]]^{1/2}$.

Theorem 2. 1) In order that there exist a G -projection of the vector g_0 onto the subspace \mathfrak{M} , it is necessary and sufficient that the functional

$$\varphi_{g_0}(f) = [g_0, f]$$

be continuous on \mathfrak{M} in the topology determined by the seminorm $|f|_{\mathfrak{M}}$ (i.e., from $\lim |f_n|_{\mathfrak{M}} = 0$ ($f_n \in \mathfrak{M}$) there follows the equality $\lim \varphi_{g_0}(f_n) = 0$).

- 2) In order that every vector in \mathfrak{H} have at least one G -projection onto \mathfrak{M} , it is necessary and sufficient that the subspace \mathfrak{M} be relatively regular.

In the proof of this theorem the following auxiliary proposition is used:

Let A and B be bounded linear operators in \mathfrak{H} . Then, in order that

$$\mathfrak{R}(A) \subseteq \mathfrak{R}(B),$$

it is necessary and sufficient that

$$\|A^*h\| \leq \alpha \|Bh\|$$

for some $\alpha > 0$ for all $h \in \mathfrak{H}$.

2. Suppose now that the functional $[f, g]$ is Hermitian-symmetric, i.e., that G is a self-adjoint operator. In this case the condition for the existence of a G -projection of a vector onto a subspace can be formulated in another form. Let $E_t^{(\mathfrak{M})}$ ($a \leq t \leq b$) be the resolution of the identity of the operator $G_{\mathfrak{M}}$ ($E_a^{(\mathfrak{M})} = 0, E_b^{(\mathfrak{M})} = P_{\mathfrak{M}}$). Then, on the basis of 2°, for the existence of $Q_{\mathfrak{M}}f_0$ it is necessary and sufficient that

$$\int_a^b \frac{1}{t^2} d\|E_t^{(\mathfrak{M})} G f_0\|^2 < \infty.$$

We have obtained a criterion for the existence of a G -projection, found by I. S. Louhivaara (2) and generalizing an earlier result of R. Nevanlinna (1).

Let us call the set \mathfrak{N} of vectors of \mathfrak{H} that are G -orthogonal to $\mathfrak{M} \subset \mathfrak{H}$ the G -orthogonal complement of the subspace \mathfrak{M} .

The following proposition follows easily from Theorem 2:

Theorem 3. *In order that the equality*

$$\mathfrak{M} \dot{+} \mathfrak{N} = \mathfrak{H}$$

hold, it is necessary and sufficient that \mathfrak{M} be nondegenerate and relatively regular. In this case \mathfrak{N} is also relatively regular, and if \mathfrak{H} is nondegenerate, then \mathfrak{N}^ is nondegenerate as well.*

3. Let us consider the special case in which the Gram operator G of the functional $[f, g]$ has the form $G = J = P_+ - P_-$ (P_+, P_- are orthogonal projectors, $P_+P_- = 0, P_+ + P_- = I$). Put $\mathfrak{H}_+ = P_+\mathfrak{H}, \mathfrak{H}_- = P_-\mathfrak{H}$.

Let \mathfrak{M} be a certain lineal in \mathfrak{H} . If there exists a linear operator K mapping $\mathfrak{M}^{(+)} = P_+ \mathfrak{M}$ onto $\mathfrak{M}^{(-)} = P_- \mathfrak{M}$ (or $\mathfrak{M}^{(-)}$ onto $\mathfrak{M}^{(+)}$) and having the lineal \mathfrak{M} as its graph, then we shall call K the angular operator of the lineal \mathfrak{M} with respect to \mathfrak{H}_+ (with respect to \mathfrak{H}_-) (cf. (7)).

Theorem 4. *In order that a lineal $\mathfrak{M} \subset \mathfrak{H}$ be a nonnegative subspace (4), it is necessary and sufficient that the angular operator K of the lineal \mathfrak{M} with respect to \mathfrak{H}_+ exist and be nonexpanding on $\mathfrak{M}^{(+)}$ ($\|K\| \leq 1$). Moreover, \mathfrak{M} is positive if and only if K is a contraction on $\mathfrak{M}^{(+)}$ ($\|Kf\| < \|f\|$ for $0 \neq f \in \mathfrak{M}^{(+)}$), null when K is isometric on $\mathfrak{M}^{(+)}$ ($\|Kf\| = \|f\|$, $f \in \mathfrak{M}^{(+)}$), and regular when $\|K\| < 1$.*

A theorem for nonpositive subspaces is formulated in a completely analogous way.

Since every subspace $\mathfrak{M} \subset \mathfrak{H}$ admits a unique decomposition

$$\mathfrak{M} = \mathfrak{M}_+ \oplus \mathfrak{M}_- \oplus \mathfrak{M}_0,$$

where \mathfrak{M}_+ , \mathfrak{M}_- , \mathfrak{M}_0 are, respectively, positive, negative, and null subspaces that are mutually orthogonal and J -orthogonal (see, for example, (*)), Theorem 4 makes it possible to give a description of all regular and irregular subspaces of the space \mathfrak{H} with indefinite J -metric.

From what has been said above it follows that *in order that every nondegenerate subspace in \mathfrak{H} be regular, it is necessary and sufficient that the dimension of at least one of the subspaces \mathfrak{H}_+ and \mathfrak{H}_- be finite* (the sufficiency of this condition was proved in (*)).

Hence, and from Theorem 2, there follow results due to L. S. Pontryagin ⁽³⁾ concerning the existence of J -projections onto a nondegenerate subspace in the case $\dim \mathfrak{H}_- < \infty$, and also the fact that, when $\dim \mathfrak{H}_+ = \dim \mathfrak{H}_- = \infty$, there exist in \mathfrak{H} nondegenerate subspaces \mathfrak{M} such that J -projections onto \mathfrak{M} do not exist for all vectors of \mathfrak{H} . However, as is not difficult to see, the set of vectors that have a J -projection onto \mathfrak{M} is dense in \mathfrak{H} . As examples show, for $G \neq J$ this, generally speaking, is not so.

Let us note that Theorem 1 makes it possible to express, for example, the J -projector onto a maximal positive subspace \mathfrak{M} ⁽⁵⁾ in terms of the angular operator K of this subspace, namely,

$$Q_{\mathfrak{M}} = (P_+ + K)(P_+ - K^*K)^{-1}(P_+ - K^*).$$

This theorem may also find application in the calculation of the coefficients of elementary divisors in the finite-dimensional case first obtained in

* After the present article had been submitted for publication, we became acquainted with the work ⁽¹⁰⁾,

from the results of which, in particular, a proposition close to Theorem 3 follows. In the case of the multiplicative decomposition, due to V. P. Potapov,⁸ of the analytic operator-function $Y(\xi)$, which is two-sidedly J -nonexpanding⁹ for $|\xi| < 1$. This application is based on the fact that if A is the leading coefficient of the Laurent expansion of the function $Y(\xi)$ in a neighborhood of the pole ξ_0 ($|\xi_0| < 1$), then $\mathfrak{R}(A)$ is a proper negative subspace.

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