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Abstract

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MATHEMATICS

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ON THE PERTURBATION OF THE SPECTRUM OF A SINGULAR ELLIPTIC OPERATOR UNDER A CHANGE OF THE BOUNDARY AND OF THE BOUNDARY CONDITIONS

(Presented by Academician V. I. Smirnov, 4 XI 1960)

The question to which the present note is devoted has been investigated by various methods⁽¹⁻³⁾. In particular, in work⁽³⁾, for certain classes of semibounded singular boundary-value problems of second order it was shown that regular changes of the boundary and of the boundary conditions outside a neighborhood of the singular points of the problem correspond to completely continuous perturbations of the inverse operator. Hence, and from Weyl's theorem on completely continuous perturbations of a self-adjoint operator, there immediately followed the invariance of the cluster spectrum* of the problem in such cases. The method used in⁽³⁾ was based on certain results of the author in the theory of extensions of semibounded operators^(5,6). Using the same method, we refine here the results of⁽³⁾ in the following directions: a) the restrictions indicated in⁽³⁾ on the growth of the coefficients of the differential expression in a neighborhood of the singular point of the problem are removed; b) the rate of decrease of the eigenvalues of the difference of the inverse operators is estimated; c) analogous estimates are obtained for the difference of integral powers of the inverse operators. In particular, it is established that the difference of sufficiently high powers of the inverse operators has a finite absolute trace. The last result makes it possible to use the Rosenblum-Kato theorem⁽⁷⁻⁹⁾, see also^(10,11), on the preservation, up to unitary equivalence, of the absolutely continuous part of a self-adjoint operator under perturbation by an operator with finite absolute trace.** Thus, in a number of cases a change of the boundary and of the boundary conditions leaves unchanged (up to unitary equivalence) the absolutely continuous part of the operator of the boundary-value problem. Below, for definiteness, we give the corresponding results for an elliptic operator of second order in the exterior of a bounded domain.

In the m -dimensional Euclidean space E_m ($m \geq 2$), consider the self-adjoint elliptic differential expression

$$\mathcal{L}u = - \sum_{i,j=1}^m \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + c(x)u. \quad (1)$$

The coefficients $a_{ij}(x)$ are continuously differentiable; the function $c(x) \geq 1$ is measurable and bounded in every ball. Let Ω be the exterior of a bounded domain in E_m with boundary Γ that is piecewise twice continuously differentiable (in the sense of S. L. Sobolev ⁽¹²⁾). The case $\Omega = E_m$ is not excluded. Let a bounded measurable function $\sigma(x)$ be given on a part Γ_2 of the boundary Γ . In the space $L_2(\Omega)$ we introduce for consideration the semibound—

* The cluster spectrum ⁽⁴⁾ of a self-adjoint operator includes the points of the continuous spectrum and eigenvalues of infinite multiplicity.

** For precise definitions and formulations see, for example, ⁽⁸⁾ or ⁽¹⁰⁾.

bounded quadratic form

$$\int_{\Omega} \sum_{i,j=1}^m \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} + c|u|^2 \right) dx + \int_{\Gamma_2} \sigma |u|^2 ds, \quad (2)$$

obtained by closing ⁽¹³⁾ from the original set of continuously differentiable functions that are finite in a neighborhood of infinity and of the surface $\Gamma_1 = \Gamma - \Gamma_2$. The form (2) generates in $L_2(\Omega)$ a semibounded self-adjoint operator S . This operator is defined by the differential expression (1) on functions satisfying, in a definite sense ^(12,14), the boundary conditions

$$u|_{\Gamma_1} = 0, \quad \frac{\partial u}{\partial \nu} + \sigma(x)u|_{\Gamma_2} = 0. \quad (3)$$

Here $\partial u / \partial \nu$ is the conormal derivative. Without restricting generality, we shall assume the operator S to be positive definite.*

Let S_1 and S_2 be two operators of the indicated type, generated by the differential expression (1) in one and the same domain Ω (the cases $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_2$ are not excluded).

Theorem 1. *The operator $S_2^{-1} - S_1^{-1}$ is completely continuous, and its eigenvalues λ_n satisfy the estimate*

$$|\lambda_n| \leq Cn^{-2/m}. \quad (4)$$

If the boundary Γ is twice continuously differentiable, then the estimate (4) may be replaced by the estimate

$$|\lambda_n| \leq Cn^{-2/(m-1)}. \quad (5)$$

Remark. The estimate (5) cannot be improved. It is attained, for example, when comparing the Dirichlet and Neumann problems for the Laplace operator in a ball. One may think that also in the case of a piecewise smooth boundary the estimate (4) can be replaced by the estimate (5).

An analogous result holds also when the boundary is changed. Obviously, it suffices to consider the case $\Gamma_1 = \Gamma$. Let T be the orthogonal sum in $L_2(E_m)$ of the operators of the first boundary-value problem for the differential expression (1) in Ω and in $E_m - \bar{\Omega}$. Consider two such operators T_1 and T_2 , corresponding to different domains Ω .

Theorem 2. *The assertions of Theorem 1 remain valid upon replacing the operator $S_2^{-1} - S_1^{-1}$ by the operator $T_2^{-1} - T_1^{-1}$.*

We note that Theorems 1 and 2 are valid also for some types of problems with an infinite boundary, as well as in the case of boundary conditions different from (3). We shall not dwell on this here.

Already the first part of Theorems 1 and 2 guarantees coincidence of the essential spectra of different operators S and T , generated by one and the same differential expression. In the case $m = 2$, the second part of Theorems 1 and 2 guarantees finiteness of the absolute trace of the operators $S_2^{-1} - S_1^{-1}$ and $T_2^{-1} - T_1^{-1}$. Hence it follows immediately:

Theorem 3. *In the case $m = 2$, a change of a twice continuously differentiable boundary or of boundary conditions of the form (3) preserves, up to unitary equivalence, the absolutely continuous part of the operator of the corresponding boundary-value problem.*

In the case $m > 2$, an analogue of Theorem 3 can be obtained on the basis of the following generalization of the estimate (4).

* This can be achieved by adding a sufficiently large constant to $c(x)$.

Theorem 4. If the coefficients $a_{ij}(x)$ and $c(x)$ have, in some neighborhood of the boundary Γ , respectively $2k - 1$ and $2k - 2$ continuous derivatives, then the eigenvalues of the operators $S_2^{-k} - S_1^{-k}$ and $T_2^{-k} - T_1^{-k}$ ($k = 1, 2, \dots$) satisfy the estimate

$$|\lambda_n| \leq Cn^{-2k/m}.$$

The last estimate leads to the following theorem:

Theorem 5. Under the hypotheses of Theorem 4, if $2k > m$, a change of the boundary or of boundary conditions of the form (3) preserves, up to unitary equivalence, the absolutely continuous part of the operator corresponding to the boundary-value problem.

Let us note that under the hypotheses of Theorem 5 the boundary Γ may be piecewise smooth. Therefore, in the case $m = 2$, Theorem 5 does not follow from Theorem 3. At the same time, Theorem 5 requires additional smoothness of the coefficients. In one important special case it is possible to dispense with this restriction.

Theorem 5a. In the case $m = 3$, the assertion of Theorem 5 remains valid for the Schrödinger operator $-\Delta u + c(x)u$, if $c(x) \geq 1$ is a measurable function, bounded in each ball.

Theorem 5a follows from Theorem 5 and from Theorem 6 below, which makes it possible to estimate the influence of a local change of the coefficients.

Theorem 6. Let S and \tilde{S} be operators in $L_2(\Omega)$ generated by two different differential expressions $\mathcal{L}u$ and $\tilde{\mathcal{L}}u$ of the form (1), under one and the same boundary condition of the form (3). If the coefficients of the operators $\mathcal{L}u$ and $\tilde{\mathcal{L}}u$ coincide outside some ball, then the operator $\tilde{S}^{-1} - S^{-1}$ is completely continuous and its eigenvalues have the estimate (4).

If, in addition, $a_{ij}(x) = \tilde{a}_{ij}(x)$ everywhere in Ω , then the eigenvalues of the operator $\tilde{S}^{-1} - S^{-1}$ have the estimate

$$|\lambda_n| \leq Cn^{-4/m}.$$

In the case $m = 3$, the last estimate guarantees the finiteness of the absolute trace of the operator $\tilde{S}^{-1} - S^{-1}$.

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