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Abstract

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ON ANALYTIC SOLUTIONS OF ONE CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

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In the present paper analytic solutions of the equation

$$\frac{\partial^r u}{\partial x^r} = a y^m \frac{\partial^n u}{\partial y^n}, \tag{1}$$

are considered, where a is a constant number; r, m, and n are natural numbers, with $m < n$. With the help of the representation, obtained in the paper, of an arbitrary analytic integral of equation (1), the Cauchy problem with respect to the variable y, as well as some new problems, is solved.

§ 1. Let $u(x, y)$ be a solution of equation (1), analytic in the bicylinder $|x| \leq R,\ |y| \leq R_1$. Then

$$\partial^r u/\partial x^r\big|_{y=0} = \partial^r u\big|_{y=0}/\partial x^r = 0,$$

whence $u(x, 0) = P_0(x)$, where $P_0(x)$ is a polynomial of degree not higher than $r - 1$; similarly, the functions $\partial^s u/\partial y^s\big|_{y=0}$, $s = 1, 2, \dots, m-1$, are also polynomials of order $\leq r - 1$. Writing the analytic solution in the form of a power series in y and substituting into (1), we arrive at the following representation:

$$u(x, y) = \sum_{l=0}^{m-1} y^l P_l(x) + \sum_{l=m}^{n-1} y^l f_l(x) + \sum_{d=0}^{n-m-1} \sum_{k=1}^{\infty} y^{m+d+k(n-m)} \alpha_{k,d} f_{m+d}^{(kr)}(x), \tag{2}$$

where

$$\alpha_{k,d} = \frac{[(k-1)(n-m)+d]!\,[(k-2)(n-m)+d]!\,\dots\,(d)!}{[(k-1)(n-m)+n+d]!\,[(k-2)(n-m)+n+d]!\,\dots\,(n+d)!\,a^k}.$$

The functions

$$f_s(x) = \partial^s u/\partial y^s\big|_{y=0}, \quad s = m, \dots, n-1,$$

will be called the **growth functions** of the solution $u(x, y)$.

Introduce the class M_{R,R_1} of analytic functions $\varphi(z)$ as follows: 1) if $n > r$, then the class M_{R,R_1} consists of functions analytic in the disk $|z| \leq R$; 2) if $n = r$, then M_{R,R_1} contains functions analytic in the disk

$$|z| \leq R + \left[\frac{R_1^{n-m}}{|a|}\left(\frac{n-m}{r}\right)^n\right]^{1/r};$$

3) finally, if $r > n$, then the class M_{R,R_1} is the totality of all entire functions of order

$$\frac{r}{r-n}$$

and type less than

$$\left(1 - \frac{n}{r}\right)\left[\frac{|a|\left(\frac{n-m}{r}\right)^n}{R_1^{n-m}}\right]^{1/(r-n)}$$

(as well as all entire functions of order less than $\frac{r}{r-n}$).

Theorem 1. In order that a solution $u(x, y)$ of equation (1) be analytic in the bicylinder $|x| \leq R$, $|y| \leq R_1$, it is necessary and sufficient that its growth functions

$$\partial^s u/\partial y^s\big|_{y=0}, \quad s = m, \ldots, n-1,$$

belong to M_{R,R_1}.

Remark 1. The general form of an analytic solution is given by formula (2).

Remark 2. If $n > r$, then the solution will be entire in y, i.e., for any fixed x, $|x| \leq R$, $u(x, y)$ is an entire function of y. If, however, $n < r$, then $u(x, y)$ (for fixed y, $|y| \leq R_1$) will be an entire function of x.

Denote by $F_{n,r}$ the collection of all entire functions if $n \geq r$, and the set of all entire functions of growth not exceeding $\left[\frac{r}{r-n}, 0\right]$ if $n < r$.

From Theorem 1 one easily obtains:

Theorem 2. In order that a solution $u(x, y)$ of equation (1) be entire in x, y (i.e., analytic in every bicylinder), it is necessary and sufficient that its growth functions

$$\left.\frac{\partial^s u}{\partial y^s}\right|_{y=0}, \qquad s = m, \ldots, n-1,$$

belong to $F_{n,r}$.

We note that the class $F_{n,r}$ does not depend on m.

It is natural to formulate the Cauchy problem with respect to the variable y for equation (1) in the domain of analytic solutions as follows: find a solution of equation (1), analytic in some bicylinder $|x| < R$, $|y| < R_1$, such that: a) $\partial^s u/\partial y^s\big|_{y=0} = P_s(x)$, $s = 0, 1, \ldots, m-1$; b) $\partial^s u/\partial y^s\big|_{y=0} = \varphi_s(x)$, $s =$

$m, \dots, n-1$, where $P_s(x)$ are polynomials of degree less than r, and $\varphi_s(x)$ are functions analytic in a neighborhood of the point $x = 0$.

It follows from Theorem 1 that, in order that an analytic solution of the Cauchy problem exist in the bicylinder $|x| < R,\ |y| < R_1$, it is necessary and sufficient that the functions $\varphi_s(x)$, $s = m, \dots, n-1$, belong to M_{R,R_1}. The solution of the Cauchy problem will be unique in the class of integrals of equation (1) analytic in a neighborhood of $(0, 0)$.

Analytic solutions of equation (1) and the Cauchy problem with respect to y in the case $m = 0$ were studied in detail by Pólya [1]; for this case he obtained results coinciding (for $m = 0$) with Theorem 1.

§ 2. Let us pose the following problem for equation (1):

Problem C. Find an integral of equation (1), entire in (x, y), satisfying the conditions:

$$\left.\frac{\partial^s u}{\partial y^s}\right|_{y=0} = P_s(x), \qquad s = 0, 1, \dots, m-1;$$

$$u(x, k_i x) = \lambda_i(x), \qquad i = 1, 2, \dots, n-m. \tag{3}$$

In the conditions (3), $P_s(x)$ are arbitrary polynomials of degree $\leq r-1$; k_i are pairwise distinct real or complex numbers; $\lambda_i(x)$ are functions analytic in a neighborhood of $x = 0$, which must satisfy certain compatibility conditions. If by $\tilde{\lambda}_i(x)$ we denote the difference

$$\lambda_i(x) - \sum_{l=0}^{m-1} (k_i x)^l P_l(x),$$

then these necessary compatibility conditions have the form

$$\tilde{\lambda}_i^{(p)}(0) = 0, \qquad p = 0, 1, \dots, m-1; \qquad \tilde{\lambda}_i^{(s)}(0) = \sum_{j=m}^{s} a_{s,j} k_i^j,$$

$$s = m, \dots, n-2; \qquad i = 1, 2, \dots, n-m. \tag{4}$$

Theorem 3. Let $u(x, y)$ be a solution of Problem C. Then: a) the functions

$$\left.\frac{\partial^s u}{\partial y^s}\right|_{y=0}, \qquad s = 0, 1, \dots, m-1,$$

must be polynomials $P_s(x)$ of degree $< r$; b) the entire functions $\lambda_i(x)$ and the polynomials $P_s(x)$ satisfy the relations (4).

Suppose now that $r \geq 2(n-m)$. Then, conversely, for any $n-m$ distinct lines $y = k_i x$ one can specify such a subclass G of sufficiently slowly growing entire functions that, if $P_s(x)$, $s = 0, 1, \dots, m-1$, are polynomials of degree $\leq r-1$, and $\lambda_i(x)$ are entire functions from G satisfying the conditions (4), then Problem C has a solution. This solution is unique in the class of integrals, entire in x, y, of sufficiently slow growth (namely, such that their growth functions belong to G).

The class G generally depends on the numbers k_i, n, m, r and a, but each time contains the collection of all polynomials (without reducing merely to it).

If the condition $r \geq 2(n-m)$ is violated, then for the solvability of problem C it is generally necessary that the functions $\lambda_i(x)$, in addition to the conditions (4) natural for analytic solutions, also satisfy conditions of another character. Moreover, uniqueness of the solution of problem C in this case may fail even in the class of entire solutions in x, y whose growth functions are polynomials.

Let us consider problem C for some concrete equations of type (1).

1.

$$\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 u}{\partial y^2}. \tag{5}$$

Problem C consists in determining an entire in x, y integral of equation (5) under the conditions $u(x, k_1 x) = \lambda_1(x)$, $u(x, k_2 x) = \lambda_2(x)$. Condition (4) in the present case has the form $\lambda_1(0) = \lambda_2(0)$. Let $Q = \max\{x_1, x_2\}$, where x_1 is the positive root of the equation

$$\left(\operatorname{ch}\frac{|k_2|}{x} - 1\right)(x+1) = \frac{1}{c_1};$$

x_2 is the positive root of the equation

$$\left(\operatorname{ch}\frac{|k_2|}{x} - 1\right)\left(1 + \frac{1}{x}\right) = \frac{1}{c_1}; \qquad c_1 = \frac{2}{|k_1 - k_2|}\sup\{1, |k_1 k_2|, |k_2|\}$$

(it is assumed that $|k_2| \geq |k_1|$). Introduce the set R_Q of entire functions growing no faster than entire functions of order $1/2$ and type $< 2/\sqrt{Q}$. From Theorem 3 one can derive the following result:

Theorem 4. *Suppose that $\lambda_1(x), \lambda_2(x)$ are entire functions from R_Q, with $\lambda_1(0) = \lambda_2(0)$. Then there exists a solution of problem C, unique in the class of entire integrals $v(x, y)$ of equation (5) for which $v(x, 0)$, $\partial v(x, y)/\partial y\big|_{y=0}$ are entire functions from R_Q.*

2.

$$\frac{\partial^r u}{\partial x^r} = a\frac{\partial u}{\partial y}.$$

In this case one has to find an entire integral $u(x, y)$ for which $u(x, kx) = \lambda(x)$. Denote by T_k the class of entire functions of order $1 - \frac{1}{r}$ and type

$$< \frac{r}{r-1}\left[\frac{|a| \ln 2}{|k|}\right]^{1/r}.$$

Then, if $\lambda(x) \in T_k$, there exists a solution of problem C, unique in the class of entire integrals $v(x, y)$ for which the function $v(x, 0)$ belongs to T_k.

3.
$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial y^2}.$$

Here it is necessary to determine an entire solution such that $u(x, k_1 x) = \lambda_1(x)$, $u(x, k_2 x) = \lambda_2(x)$. The condition $r \geq 2(n - m)$ is not fulfilled in this case, and Theorem 3 is inapplicable. Analyzing the known formula for the general solution of the wave equation, we easily find that, for example, in the case where $k_2 = 1/k_1$, for the solvability of problem C it is necessary that the function $\Phi(x) = \lambda_2(k_1 x) - \lambda_1(x)$ be odd. If this condition is satisfied, then a solution of problem C exists and has the form

$$u(x, y) = \Phi\left(\frac{x+y}{1+k_1}\right)/2+\lambda_1\left(\frac{x+y}{1+k_1}\right)-\mu\left[(x+y)\frac{(1-k_1)}{(1+k_1)}\right]+\mu(x-y)+\Phi\left(\frac{x-y}{k_1-1}\right)/2,$$

where $\mu(x)$ is an arbitrary even function. This example shows that violation of the condition $r \geq 2(n - m)$ substantially changes the nature of the conclusions concerning solvability and uniqueness of problem C.

§ 3. In an analogous manner one investigates:

Problem C$_1$. Determine an entire integral of equation (1) under the conditions:
a) $\partial^s u/\partial y^s\big|_{y=0} = P_s(x)$, $s = 0, 1, \dots, m - 1$;
b) $\partial^s u/\partial y^s\big|_{y=k_s x} = \lambda_s(x)$, $s = m, \dots, n - 1$. In these conditions $P_s(x)$ are, as before, polynomials of degree $< r$; k_s are pairwise distinct numbers; $\lambda_s(x)$ are entire functions; no compatibility conditions are required.

If $r \geqslant 2(n-m)$, then for any $n-m$ distinct straight lines one can specify a set G_1 of sufficiently slowly increasing entire functions such that, if $\lambda_s(x) \in G_1$, then there exists a solution of problem C$_1$, unique in the class of entire integrals $v(x, y)$ whose growth functions belong to G_1. Violation of the condition $r \geqslant 2(n - m)$ leads to the same consequences as in problem C.

In both problems one can specify a metric in which there is continuous dependence of the solution on the data of the problem.

In conclusion, we note that problems C and C$_1$ resemble the problem considered by V. P. Mikhailov (^{2}) for general systems of first-order partial differential equations. At the same time, these problems, in their formulation and in the character of the results, differ from V. P. Mikhailov' s problem and cannot be reduced to it.

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References Cited

[1] E. Rothe. *Math. Zs.*, **28**, 48 (1928).
[2] V. P. Mikhailov, *Matem. sborn.*, **46** (88), No. 3 (1958).

Note: Figure translations are in progress. See original paper for figures.

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