MATHEMATICS

L. I. KAMYNIN

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. I. KAMYNIN

ON THE SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR A PARABOLIC EQUATION WITH DISCONTINUOUS COEFFICIENTS

(Presented by Academician S. L. Sobolev on 1 IV 1961)

Much attention has recently been devoted to the solution of boundary-value problems for linear parabolic equations with discontinuous coefficients ($^{3-11}$). Investigations have been carried out both for one (^{5}) and for many spatial variables ($^{3,4,6-11}$). We note that in these studies both the coefficients of the equation outside the surfaces (lines) of discontinuity and the surfaces (lines) of discontinuity themselves were assumed to be sufficiently smooth. The aim of the present work is to prove the existence of a solution of the I, II, and III boundary-value problems for a parabolic equation with discontinuous coefficients (with one spatial variable) in the case of lines of discontinuity satisfying only a Hölder condition with exponent $> 1/2$. In the proof we use results and methods of the classical papers of Gevrey (^{1}) and Holmgren (^{2}).

1. Consider, in the (x, t)-plane, the domains

$$S_T^{(i)} = \{(x,t);\ X_j(t) < x < X_{j+1}(t);\ 0 < t < T\}, \quad (i = 1, 2)$$

where $j = 1$ for $i = 1$; $j = 3$ for $i = 2$, and $\overline{S_T^{(i)}}$ is the closure of $S_T^{(i)}$. The functions $X_j(t)$ $(j = 1, 2, 3, 4)$, which determine the lateral boundaries of $S_T^{(i)}$, are always assumed to satisfy the Hölder condition

$$|X_j(t) - X_j(\bar{t})| \le K|t - \bar{t}|^{(1+\delta)/2}, \tag{1}$$

where K and δ are positive constants $(0 < \delta \le 1)$. The curves $x = X_i(t)$ $(0 \le t \le T)$ for $i = 1, 2$ or $i = 3, 4$ have no common points, while for $i = 2, 3$ they may either intersect or, in particular, completely coincide. In the domain $S_T^{(i)}$ $(i = 1, 2)$ we consider a parabolic equation of the form

$$a_i(x,t)\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} + b_i(x,t)\frac{\partial u_i}{\partial x} + c_i(x,t)u_i + f_i(x,t) \quad (i = 1, 2) \tag{2}$$

and shall seek a solution $u_i(x,t)$ of equation (2) satisfying the initial data

$$u_i(x,0) = F_i(x), \quad X_j(0) \le x \le X_{j+1}(0) \tag{3}$$

$$(j = 1 \text{ for } i = 1;\ j = 3 \text{ for } i = 2),$$

the boundary conditions

$$\frac{\partial u_i(X_j(t),t)}{\partial x} + \lambda_i(t) u_i(X_j(t),t) = \varphi_i(t), \quad 0 \le t \le T \tag{4}$$

$$(j = 1 \text{ for } i = 1;\ j = 4 \text{ for } i = 2)$$

(in particular, it may be that $\lambda_i(t) \equiv 0$) and the conditions on the lines of discontinuity $x = X_i(t)$, $i = 2, 3$:

$$\alpha_1(t)\frac{\partial u_1(X_2(t),t)}{\partial x} - \alpha_2(t)\frac{\partial u_2(X_3(t),t)}{\partial x} = h(t), \tag{5}$$

$$u_1(X_2(t),t) - u_2(X_3(t),t) = r(t), \qquad 0 \le t \le T, \tag{6}$$

provided the compatibility conditions are satisfied

$$F_i'(X_j(0)) + \lambda_i(0) F_i(X_j(0)) = \varphi_i(0) \tag{7}$$

$$(j = 1 \text{ for } i = 1;\ j = 4 \text{ for } i = 2);$$

$$\alpha_1(0) F_1'(X_2(0)) - \alpha_2(0) F_2'(X_3(0)) = h(0); \tag{8}$$

$$F_1(X_2(0)) - F_2(X_3(0)) = r(0). \tag{9}$$

One or both conditions (4), corresponding to the boundary-value problem II ($\lambda_i(t) \equiv 0$) or III, may be replaced by the conditions of boundary-value problem I

$$u_i(X_j(t),t) = \psi_i(t), \qquad 0 \le t \le T \tag{10}$$

$$(j = 1 \text{ for } i = 1;\ \text{for } i = 2,\ j = 4$$

with replacement of the compatibility conditions (7) by

$$F_i(X_j(0)) = \psi_i(0) \tag{11}$$

$$(j = 1 \text{ for } i = 1;\ j = 4 \text{ for } i = 2).$$

We shall seek a solution $u_i(x,t)$ of problem (2)–(10) satisfying equation (2) inside the domain $S_T^{(i)}$ and continuous, together with $\partial u_i/\partial x$, on $\overline{S_T^{(i)}}$.

2. Let us first consider, as (2), the heat-conduction equation of the form

$$a_i^2 \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} + f_i(x,t), \qquad (i = 1, 2). \tag{12}$$

Theorem 1. *Suppose the following conditions are fulfilled:*

a) *The functions* $\lambda_i(t)$, $\varphi_i(t)$, $\psi_i(t)$, $\alpha_i(t)$ $(i = 1,2)$, $h(t)$, $r(t)$ *are continuous for* $0 \le t \le T$*; the functions* $F_i(x)$ *are continuous together with* $F_i'(x)$ $(i = 1,2)$ *on the intervals* $[X_1(0), X_2(0)]$, $[X_3(0), X_4(0)]$, *respectively; the functions* $f_i(x,t)$ *are continuous in* (x,t) *in* $\overline{S_T^{(i)}}$*, and for* $f_i(x,t)$ *one of the conditions (A) of Gevrey is fulfilled ((*[1]*, pp. 350, 351)), for example,*

$$|f_i(x,t_1) - f_i(x,t_2)| \le k|t_1 - t_2|^\gamma \qquad (\gamma > 0 \text{ constant}). \tag{13}$$

b) *The functions* $X_i(t)$ $(i = 1,2,3,4)$ *satisfy on* $[0,T]$ *a Hölder condition with exponent* $> 1/2$ *(1).*

c) *The compatibility conditions (8), (9), and (7) (or (11)) are fulfilled.*

d) a_1, a_2 *are positive constants, and*

$$a_1\alpha_2(t) + a_2\alpha_1(t) \ne 0, \qquad 0 \le t \le T.$$

e) $r(t)$, $\psi_i(t)$ *satisfy a Hölder condition with exponent* $> 1/2$.

Then there exists a solution $u_i(x,t)$ $(i = 1,2)$ *of problem (12), (3), (4) (or (10)), (5), (6), continuous, together with* $\partial u_i/\partial x$, *on* $\overline{S_T^{(i)}}$.

3. Let us now consider, instead of (12), the more general equation

$$a_i^2 \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} + b_i(x,t)\frac{\partial u_i}{\partial x} + c_i(x,t)u_i + f_i(x,t) \qquad (i = 1, 2). \tag{14}$$

Theorem 2. *Suppose the conditions a)–e) of Theorem 1 are fulfilled and, in addition:*

e) *The functions* $b_i(x,t)$, $c_i(x,t)$ *are continuous in* (x,t) *in* $\overline{S_T^{(i)}}$ *and satisfy one of the conditions (A) of Gevrey, for example the Hölder condition with nonzero exponent in* t *of the form (13).*

Then there exists a solution $u_i(x,t)$ $(i=1,2)$ of problem (14), (3), (4) (or (10), (5), (6)), continuous together with $\partial u_i/\partial x$ on $\overline{S}_T^{(i)}$.

4. Let us return to the general equation (2).

Theorem 3. Suppose that conditions a)—c), d), and e) of Theorem 2 are satisfied and, in addition:

) The function $a_i(x,t)$ is continuous in $\overline{S}_T^{(i)}$ together with the derivatives $\partial a_i(x,t)/\partial x$, $\partial a_i(x,t)/\partial t$, and $\partial a_i(x,t)/\partial x$, $\partial a_i(x,t)/\partial t$ satisfy one of Gevrey' s conditions (A) (for example, the Gel′der condition of the form (13)).

) In the domain $\overline{S}_T^{(i)}$, $a_i(x,t) \geq a_i^2 > 0$, where a_i is a constant, and

$$\sqrt{a_1(X_2(t),t)}\, a_2(t) + \sqrt{a_2(X_3(t),t)}\, a_1(t) \neq 0, \qquad 0 \leq t \leq T.$$

Then there exists a solution of problem (2), (3), (4) (or (10), (5), (6)), continuous together with $\partial u_i/\partial x$ on $\overline{S}_T^i$.

Remark 1. Theorems 1—3 are valid for any finite number of domains of the form $S_T^{(i)}$ $(i=1,2,\dots,k)$.

Remark 2. If $X_i(t)$ $(i=1,2,3,4)$ have derivatives $X_i'(t)$ satisfying the Gel′der condition (1) $(X_2(t) \equiv X_3(t))$, then a result analogous to Theorem 3 was proved by A. A. Samarskii (^{5}).

5. We shall briefly outline the proofs of Theorems 1—3.

Proof of Theorem 1. First the problem is reduced to homogeneous equations (12) $(f_i(x,t) \equiv 0)$, zero initial data (3) $(F_i(x) \equiv 0)$, and homogeneous boundary conditions (4) (or (10), (5), (6)) $(\varphi_i(t) \equiv 0,\ \psi_i(t) \equiv 0,\ h(t) \equiv 0,\ r(t) \equiv 0)$. Then the solution in each domain $S_T^{(i)}$ $(i=1,2)$ is sought in the form of a sum of heat potentials of a simple layer. For the 4 unknown densities of the potentials, by virtue of conditions (4) (or (7)), (5), (6), four integral equations of Volterra type are derived; moreover, to conditions (4), (5) there correspond Volterra equations of the 2nd kind, and to conditions (7), (6) Volterra equations of the 1st kind, which, by means of Holmgren' s method (^{2}), can, when conditions),),) of Theorem 1 are fulfilled, be replaced by equivalent Volterra equations of the 2nd kind. In the end one obtains a system of 4 singular integral equations of Volterra type of the 2nd kind, having kernels with weak singularities whose order does not exceed $1-\delta/2$. This system always has a unique solution in the class of continuous functions.

Proof of Theorem 2. The problem for equation (14) is reduced to the solution of an integro-differential equation of the form

$$u_i(x,t) = -\iint_{S_t^{(i)}} \left[b_i(\xi,\tau)\frac{\partial u_i(\xi,\tau)}{\partial \xi} + c_i(\xi,\tau)u_i(\xi,\tau) \right] U_i(\xi,\tau;x,t)\,d\xi\,d\tau +$$

$$+ v_i(x,t) - \overline{v}_i(x,t) \qquad (i = 1,2), \tag{15}$$

where

$$U_i(\xi,\tau;x,t) = \frac{1}{2a_i\sqrt{\pi(t-\tau)}} \exp\left\{ -\frac{(x-\xi)^2}{4a_i^2(t-\tau)} \right\}$$

is the fundamental solution of the heat equation (12); $v_i(x,t)$ is the solution constructed in Theorem 1; $\overline{v}_i(x,t)$ is the solution of the homogeneous equation (12) ($f_i(x,t) \equiv 0$), chosen so that conditions (3), (4) (or (7)), (5), (6) are satisfied. Equation (15) is solved by the method of successive approximations, using Gevrey estimates: $((^1), pp. 356, 358)$.

Proof of Theorem 3. The problem for equation (2) can be reduced to the problem for equation (14) considered in Theorem 2, if

in each of the regions $S_T^{(i)}$ make the substitution $\bar{t} = t$:

$$y = \int_{X_i(0)}^{x} \frac{d\xi}{\sqrt{a_i(\xi,t)}}$$

($j = 1$ for $i = 1$; $j = 3$ for $i = 2$) (see (1), p. 370); in this case the curves $x = X_i(t)$ will pass into the curves $y = Y_i(\bar{t})$, where the functions $Y_i(\bar{t})$ will satisfy the Hölder condition (1) with the same exponent $(1+\delta)/2$.

Moscow State University
named after M. V. Lomonosov

Received
31 III 1961

REFERENCES

[1] M. Gevrey, J. Math. pures et appl., **9**, 305 (1913).
[2] E. Holmgren, Ark. f. Mat., Astr. och Fys., **3**, H. 2, No. 12 (1907).
[3] O. A. Ladyzhenskaya, DAN, **96**, No. 3 (1954).
[4] E. I. Kim, Doctoral dissertation, *On one class of singular integral equations and certain problems of heat conduction for piecewise-homogeneous media*, V. A. Steklov Mathematical Institute, USSR Academy of Sciences, 1956.
[5] A. A. Samarskii, DAN, **121**, 225 (1958).
[6] O. A. Oleinik, DAN, **124**, No. 6, 1219 (1959).
[7] O. A. Oleinik, UMN, **14**, issue 5 (89), 164 (1959).

[8] L. I. Kamynin, V. N. Maslennikova, DAN, **133**, No. 5, 1003 (1960).
[9] I. V. Girsanov, DAN, **135**, No. 6, 1311 (1960).
[10] O. A. Oleinik, Izv. AN SSSR, ser. matem., **25**, No. 1, 3 (1961).
[11] L. I. Kamynin, V. N. Maslennikova, Sibirsk. matem. zhurn., **2**, No. 3 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.