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Abstract

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DIVERGENT FOURIER SERIES OF CLASS L^p $(p \geq 2)$

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Let $\{\varphi_k(x)\}$ $(k = 1, 2, ...)$ be an orthonormal system on the interval $[a, b]$. Many authors have studied the question of constructing divergent (on one set or another) Fourier series of the form

$$\sum_{k=1}^{\infty} c_k \varphi_k(x). \tag{1}$$

A. N. Kolmogorov (3) was the first to construct a fundamentally important example of a function $f(x) \in L(0, 2\pi)$ whose trigonometric Fourier series diverges almost everywhere on $[0, 2\pi]$. Somewhat later he also constructed an example of a trigonometric Fourier series that diverges everywhere (4) (see also (15), p. 175). Subsequently these results were generalized in various directions. In (10) we gave a survey of the main results concerning the construction of divergent trigonometric Fourier series of class L possessing one or another additional property.

As for general orthogonal series, in this direction fundamental results were obtained by D. E. Menshov. Thus, in particular, he was the first to construct (7) an example of an everywhere divergent orthogonal Fourier series of class L^2, i.e. an example of an everywhere divergent series of the form (1) with $\sum c_k^2 < \infty$ (see also (2), p. 537). Later these results were generalized by Tandori (9) (see also (1)).

In connection with what has been said, the question naturally arises of the existence of trigonometric series

$$\sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad \text{with} \quad \sum_{k=1}^{\infty} (a_k^2 + b_k^2) < \infty, \tag{2}$$

which would diverge on a set of positive (or full) measure. This question was posed by N. N. Luzin (6), p. 219, 45 years ago; namely, he speaks of the probability that every trigonometric Fourier series of class L^2 converges almost everywhere. At present there is no answer to the indicated question. In exactly the same position is the question concerning the convergence of trigonometric Fourier series of continuous functions and of functions of class L^p with $p > 1$.

Further, in a paper of A. N. Kolmogorov and D. E. Menshov (^{5}) the following assertion was formulated (belonging to A. N. Kolmogorov):

There exists a function $f(x) \in L^2(0, 2\pi)$ *such that the terms of its trigonometric Fourier series can be rearranged so that the newly obtained series again diverges almost everywhere on* $[0, 2\pi]$.

Since we know nothing about the convergence or divergence of series of the form (2), the indicated assertion is important and fundamental. But the assertion made by A. N. Kolmogorov almost 35 years ago was given without proof. Only very recently did a paper by Zagorskii (^{14}) appear in which, although without a detailed proof,

a scheme is given for the construction of the indicated series. Namely, Zagorskii describes how the trigonometric system must be rearranged in order that the corresponding divergent series can be constructed.

At the present time, on the basis of Zagorskii' s published work (^{14}), one may already regard A. N. Kolmogorov' s assertion as restored. Thus, the following is true.

Theorem A. There exist almost everywhere divergent Fourier series of class L^2 with respect to a rearranged trigonometric system.

Starting from Theorem A, we proved ((^{11}), p. 828) that the following holds.

Theorem 1. There exists a function $F(x) \in L^p(0, 2\pi)$ for all $p > 0$ such that, for any summability method T^*, the terms of the trigonometric Fourier series of the function $F(x)$ can be rearranged so that the newly obtained series is not summable by the method T^* almost everywhere on $[0, 2\pi]$, i.e. the Fourier series of the function $F(x)$ with respect to the rearranged trigonometric system will be almost everywhere not summable by the method T^*.

Let us note that the class of methods T^* contains the entire class of regular Toeplitz methods.

In connection with Theorem 1 it is useful to note the following two circumstances:

1) Directly from the course of the proof of Theorem 1 ((^{11}), p. 829) it is seen that the trigonometric Fourier series (in the proper order) of the function $F(x)$ may be considered convergent almost everywhere on $[0, 2\pi]$, and that

$$\int_0^{2\pi} e^{CF^2(x)}\, dx < \infty \tag{3}$$

for any constant $C > 0$. It is obvious that fulfillment of inequality (3) is a stronger assertion than the fact that $F(x) \in L^p(0, 2\pi)$ for all $p > 0$.

2) Trigonometric Fourier series of the form (2) are obliged to be summable almost everywhere, for example, by Cesàro methods (C, α) with $\alpha > 0$ (([15]), p. 54), whereas rearranged Fourier series, even of classes L^p with arbitrary $p > 2$, are not obliged to possess such a property, whatever summability methods we take.

The divergence (and non-summability) of rearranged trigonometric Fourier series of class L^p with $1 < p < 2$ was investigated in detail by us in the works ([11,]).

In the present work we shall give assertions concerning series with respect to the Walsh system $\{\omega_n(x)\}$ (([15]), p. 18 and ([2]), p. 155) and with respect to the Haar system $\{\chi_n^{(k)}(x)\}$ (([2]), p. 57). Below we shall need an assertion that was proved by the author (([11]), p. 816). Namely, the following is true.

Theorem B. Let the functions $f_k(x)$ $(k = 1, 2, ...)$ be defined on a measurable set E and

$$\int_E |f_k(x)|\, dx \leq C \qquad (k = 1, 2, ...),$$

where C is some constant. Then, if the series

$$\sum_{k=1}^{\infty} c_k f_k(x) \qquad \left(\lim_{k\to\infty} |c_k| = 0\right) \tag{4}$$

diverges almost everywhere on E, then for any summability method T^* the terms of the series (4) can be rearranged so that the newly obtained series is not summable by T^* almost everywhere on E.

Modifying Zagorskii' s construction ([14]) in the corresponding way and using Theorem B, one can prove that the following two theorems are true.

Theorem 2. There exists a function $F(x) \in L^p(0, 1)$ for all $p > 0$ such that, for every summability method T^*, some rearranged Fourier series of the function $F(x)$ with respect to the Walsh system $\{\omega_n(x)\}$ is not summable by the method T^* almost everywhere on $[0, 1]$.

Since Walsh functions $\omega_n(x)$ take, almost everywhere, only the values ±1, Theorem 2 is a strengthening of the theorem of Kolmogorov—Menshov (([5]), Theorem 1), in which it is asserted that there exists an orthogonal series (1) from the class L^2 that diverges everywhere and in which $\varphi_k(x)$ take only the values ± 1.

Theorem 3. There exists a function $F(x) \in L^2(0, 1)$ such that, for every method T^*, some rearranged Fourier series of the function $F(x)$ with respect to the Haar system $\{X_n^{(k)}(x)\}$ is not summable by the method T^* almost everywhere on $[0, 1]$.

From Theorem 3 there follows a number of statements that are also of independent interest. We first recall and introduce definitions:

Definition 1. A functional series is called **weakly unconditionally convergent** almost everywhere on E if it converges almost everywhere on E under any weak rearrangement of its terms.*

We note that Definition 1 was introduced by us in the paper (^{12}).

Definition 2. Let $\{\varphi_k(x)\}$ be an orthonormal system on $[0, 1]$. Then the system $\{\varphi_k(x)\}$ is called a **system of convergence** if every series

$$\sum c_k \varphi_k(x) \quad \text{with} \quad \sum c_k^2 < \infty \tag{5}$$

converges almost everywhere on $[0, 1]$.

Definition 3. A system $\{\varphi_k\}$ is called a **system of unconditional convergence** (**weak unconditional convergence**) if every series of the form (5) converges unconditionally (respectively, weakly unconditionally) almost everywhere on $[0, 1]$.

It is easy to prove that every system of convergence is a system of weak unconditional convergence.

It is known ((^{2}), p. 143) that the Haar system $\{X_n^{(k)}(x)\}$ is a system of convergence and, consequently, a system of weak unconditional convergence. Theorem 3 shows that the Haar system is not a system of unconditional convergence. Thus we obtain:

Theorem 4. There exists a complete orthonormal system which is a system of convergence (a system of weak unconditional convergence) and is not a system of unconditional convergence.

Theorem 5. There exists a functional series

$$\sum_{n=1}^{\infty} f_n(x) \quad (x \in [0, 1],\ f_n(x) \text{ measurable}), \tag{6}$$

for which every subseries

$$\sum_{k=1}^{\infty} f_{n_k}(x) \quad (n_1 < n_2 < ...)$$

converges almost everywhere on $[0, 1]$, and yet the series (6), after some rearrangement of its terms, diverges almost everywhere on $[0, 1]$.

It is useful to note that if we consider series

$$\sum_{n=1}^{\infty} f_n, \tag{7}$$

where f_n are elements of a Banach space B, then, as Orlicz proved ((^{2}), p. 42), convergence (in the norm of B) of all subseries of the series (7) is equivalent to convergence of the series (7) under any order of arrangement of the terms. A statement of the same type was proved by Orlicz (^{8}) also for the case when convergence in measure of functional series is considered. Thus, Theorem 5 indicates that, in the present question, the case of convergence almost everywhere is fundamentally different from the cases of convergence in measure or in norm.

Further, Steinhaus (^{13}) proved that the Rademacher system ((^{2}), pp. 55, 147) is a system of unconditional convergence. But the Rademacher system is not complete. The question of whether there exists a **complete** orthonormal system of unconditional convergence almost everywhere has not at present been resolved. We might suppose that the Haar system is such a system, since it is a system of weak unconditional convergence. But Theorem 3 shows that this is not so.

Let us formulate two more theorems concerning the same circle of questions.

Theorem 6. *There exists an orthogonal series*

$$\sum_{k=1}^{\infty} a_k \varphi_k(x) \quad \text{with} \quad \sum_{k=1}^{\infty} a_k^2 < \infty \quad (|\varphi_k(x)| \leq C,\ C = \text{const}), \tag{8}$$

which converges everywhere on $[0,1]$ *and, nevertheless, is not weakly unconditionally convergent. Moreover, the series (8), after some weak permutation, diverges almost everywhere on* $[0,1]$.

Theorem 7. *If the series (1), where* $\{\varphi_k\}$ *is a complete orthonormal system (or* $\{\varphi_k\}$ *is an orthonormal system bounded in the aggregate), is weakly unconditionally convergent almost everywhere on a set* E *with* $mE > 0$ *(respectively on* $[0,1]$*), then*

$$\lim_{k\to\infty} |c_k| = 0 \quad \left(\text{respectively } \sum_{k=1}^{\infty} c_k^2 < \infty\right). \tag{9}$$

An assertion of this type, like Theorem 7, for the case of unconditional convergence was proved by us earlier ((11^6), pp. 832, 833). We note that, for a preassigned orthonormal system, the conditions (9), generally speaking, are not sufficient for weak unconditional convergence of the series (1). If, however, we

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* A rearranged natural series of numbers is called a weakly rearranged natural series of numbers if it is split i

consider the whole class of orthonormal systems, then these conditions become sufficient in the following sense: if $\lim_{k\to\infty} |c_k| = 0$, then there exists a complete orthonormal system $\{\varphi_k\}$ (for example, a rearranged Haar system) such that the series (1) is weakly unconditionally convergent almost everywhere on $[0, 1]$. If $\sum c_k^2 < \infty$, then there exists an orthonormal system $\{\varphi_k\}$, bounded in the aggregate (for example, a rearranged Walsh system, the Rademacher system, etc.), such that the series (1) is weakly unconditionally convergent almost everywhere on $[0, 1]$.

Finally, we give one more theorem, having a hypothetical character and relating to trigonometric series in the natural order.

Theorem 8. *If there exists a function $f(x)$, continuous on $[0, 2\pi]$ (or $f \in L^p$ for some $p > 1$), whose trigonometric Fourier series diverges on a set of positive measure, then there exists a continuous function $F(x)$ (respectively $F \in L^p$) whose Fourier series diverges without bound almost everywhere on $(-\infty, +\infty)$.*

This theorem shows that either all trigonometric Fourier series of class C (of class L^p) converge almost everywhere, or else there exist Fourier series from class C (respectively from L^p) which diverge without bound almost everywhere.

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Note: Figure translations are in progress. See original paper for figures.

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