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Abstract

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MATHEMATICS

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ADDITIVE FUNCTIONALS OF A MULTIDIMENSIONAL WIENER PROCESS

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Let $X = (x_t, \xi, \mathcal{M}_t, P_x)$ be a homogeneous Markov process (for the terminology and notation used, see (^{1})). In this note, by a **functional** we shall mean an almost homogeneous, almost additive functional of X, defined for $0 \leq s \leq t < \infty$, i.e., a function $\varphi_t^s(\omega)$ satisfying the following conditions:

A. φ_t^s is measurable in ω with respect to the σ-algebra $\mathcal{N}_t^s$ generated by the events $\{x_u \in \Gamma\}$, $0 \leq u \leq t$.

B. $\varphi_u^s = \varphi_t^s + \varphi_u^t$ a.s. (almost surely) (P_x).

C. $\theta_h \varphi_t^s = \varphi_{t+h}^{s+h}$ a.s. (P_x).

In the survey report (^{2}) the problem was posed of describing all functionals of a given process. Most of the results obtained so far concern the case of nonnegative functionals. In the present note we study functionals having zero mathematical expectation.

Let X be an N-dimensional Wiener process. We shall not introduce any special notation for vectors and scalar product. If a vector function f is such that, a.s. (P_x),

$$\int_0^t f^2(x_u)\, du < \infty \tag{1}$$

for all t, x, then one can define the stochastic integral $\int_s^t f(x_u)\, dx_u$, which is a functional of X continuous in s and t. If the mathematical expectation of expression (1) is finite, then the mathematical expectation of this functional is equal to 0 (a small refinement of Theorem 2 of (^{3})).

Theorem. Let φ_t^s be a functional of X such that $M_x \varphi_t^s = 0$. Then there exists a Borel function f such that, a.s. (P_x),

$$\varphi_t^s = \int_s^t f(x_u)\, dx_u. \tag{2}$$

We shall carry out the proof in the substantially simpler case when

$$M_x(\varphi_t^s)^2 < C_t < \infty, \qquad C_t \to 0 \quad \text{as } t \to 0. \tag{3}$$

Define a certain new functional with vector values. To this end introduce the notation $x_t^s = x_t - x_s$; consider some partition $s = t_0 < t_1 < \dots < t_n = t$ of the interval from s to t, and put

$$[\varphi\varphi]_t^s = \sum_{i=1}^{n} (\varphi_{t_i}^{t_{i-1}})^2, \qquad [xx]_t^s = \sum_{i=1}^{n} (x_{t_i}^{t_{i-1}})^2, \qquad [\varphi x]_t^s = \sum_{i=1}^{n} \varphi_{t_i}^{t_{i-1}} x_{t_i}^{t_{i-1}}. \tag{4}$$

It is easy to show that the mathematical expectations of the sums (4) do not depend on the partition. Let us prove this, for example, for the last sum. We have:

$$M_x \varphi_t^s x_t^s = M_x \left[\sum_{i=1}^{n} \varphi_{t_i}^{t_{i-1}}\right] \left[\sum_{i=1}^{n} x_{t_i}^{t_{i-1}}\right] = \sum_{i=1}^{n} M_x \varphi_{t_i}^{t_{i-1}} x_{t_i}^{t_{i-1}} +$$

$$+ \sum_{i<j} M_x \left[\varphi_{t_i}^{t_{i-1}} M_x\{x_{t_j}^{t_{j-1}} / \mathcal{M}_{t_{j-1}}\}\right] + \sum_{i<j} M_x \left[x_{t_i}^{t_{i-1}} M_x\{\varphi_{t_j}^{t_{j-1}} / \mathcal{M}_{t_{j-1}}\}\right] = M_x[\varphi x]_t^s.$$

The expression $[\varphi x]_t^s$ has a limit in mean square as the partition is refined. To prove this, one considers the difference between the sums corresponding to two partitions, one of which is obtained from the other by adding new points. After transformations, which we do not give here and which use as the principal device the representation of a mathematical expectation in the form of the mathematical expectation of a conditional mathematical expectation, we obtain that the mathematical expectation of the square of this difference does not exceed $8NC_\Delta(t-s)$, where Δ is the length of the largest interval of the coarser partition. Hence it follows that there exists

$$\alpha_t^s = \text{l.i.m.}\, [\varphi x]_t^s. \tag{5}$$

This is a functional of X; it is almost homogeneous, since applying the operator θ_h to $[\varphi x]_t^s$ gives a sum of the same form for α_{t+h}^{s+h}; it is almost additive, since $[\varphi x]_u^s = [\varphi x]_t^s + [\varphi x]_u^t$ for partitions containing the point t.

Let us prove that α_t^s can be chosen so that α_t^0 is, with probability 1, absolutely continuous in t, with derivative square-integrable on every finite interval. It is known ((4), p. 85) that, in order that a function $F(t)$ be the integral of a function square-integrable from 0 to T, it is necessary and sufficient that the sums

$$\sum_{i=1}^{n} \frac{(F(T_i) - F(T_{i-1}))^2}{T_i - T_{i-1}},$$

where $0 = T_0 < T_1 < \cdots < T_n = T$, be bounded. Let us estimate these sums for the function α_t^0, which are almost surely equal to

$$\sum_{i=1}^{n} \frac{(\alpha_{T_i}^{T_{i-1}})^2}{T_i - T_{i-1}},$$

for rational T_i. It is known that

$$T_i - T_{i-1} = \frac{1}{N} \,\text{l.i.m.}\, [xx]_{T_i}^{T_{i-1}}$$

as the partition is refined. Choose a sequence of partitions so that this limit and the limit (5) exist with probability 1 and so that every rational point is a point of some partition. Almost surely,

$$\sum_{i=1}^{n} \frac{(\alpha_{T_i}^{T_{i-1}})^2}{T_i - T_{i-1}} = N \lim \sum_{i=1}^{n} \frac{\left([\varphi x]_{T_i}^{T_{i-1}}\right)^2}{[xx]_{T_i}^{T_{i-1}}} \leq \lim \sum_{i=1}^{n} [\varphi\varphi]_{T_i}^{T_{i-1}} = N \lim [\varphi\varphi]_T^0. \quad (6)$$

The mathematical expectation of this expression—the lower limit of a nonnegative quantity—does not exceed the limit of the mathematical expectation, which exists and is equal to $NM_x(\varphi_T^0)^2$. Therefore, almost surely, α_t^0 can be extended from rational values of t in an absolutely continuous way. From estimate (6) follows the stochastic continuity of α_t^0, and therefore the new α_t^0 will also be a version of the limit (5).

We shall now show that $\alpha_t^s = \int_s^t f(x_u)\, du$ a.s. (P_x). Put

$$\xi = \overline{\lim_{n\to\infty}}\, n\alpha_{1/n}^0.$$

This random variable is measurable with respect to the σ-algebra $\mathcal{N}_{+0}$ and, according to the zero-one law ((1), p. 153), assumes with P_x-probability 1 only one⋯

value, which we shall also take as $f(x)$; the function f is Borel. This follows from the fact that $f(x) = M_x\xi$. It remains to prove that with probability 1, for almost all t,

$$\lim_{n\to\infty} n\left(a_{t+1/n}^0 - a_t^0\right) = f(x_t).$$

Since both these quantities are measurable in (t, ω), by Fubini' s theorem it suffices to prove that, for almost all t, with probability 1,

$$\lim_{n\to\infty} n\left(a^0_{t+1/n} - a^0_t\right) = f(x_t).$$

But, for all t, these quantities are equal with probability 1 to $\theta_t \xi$.

Estimate (6) shows that, with probability 1, the function $f(x_t)$ is square-integrable on any finite interval, and moreover

$$M_x \int_0^T f^2(x_t)\, dt \leq NC_T < \infty,$$

therefore $\int_s^t f(x_u)\, dx_u$ is defined. We shall show that the functional

$$\widetilde{\varphi}^s_t \equiv \varphi^s_t - \int_s^t f(x_u)\, dx_u = 0 \quad \text{a.s. } (P_x).$$

In doing so we shall use only the fact that $M_x(\widetilde{\varphi}^s_t)^2$ exists; $M_x \widetilde{\varphi}^s_t = 0$; $M_x x^s_t \widetilde{\varphi}^s_t = 0$. The last fact follows from

$$M_x x^s_t \int_s^t f(x_u)\, dx_u = M_x \int_s^t f(x_u)\, du = M_x a^s_t = M_x x^s_t \varphi^s_t.$$

In the proof we shall need the following lemma, due to Meyer:

Lemma. Let $\widetilde{\varphi}^s_t$ be a functional of the Markov process X. If for all t and x, and for every bounded Borel function F, the equality

$$M_x \widetilde{\varphi}^0_t F(x_t) = 0, \tag{7}$$

holds, then $\widetilde{\varphi}^s_t = 0$ a.s. (P_x).

The proof consists in verifying the relation

$$M_x F_1(x_{t_1}) F_2(x_{t_2}) \cdots F_n(x_{t_n}) \widetilde{\varphi}^0_t = 0;$$

this is sufficient, since $\widetilde{\varphi}^0_t$ is measurable with respect to the σ-algebra $\mathcal{N}_t$.

For a function F that is bounded and continuous together with its first and second partial derivatives, by the change-of-variables formula in the stochastic integral (see (5)) we have a.s. (P_x):

$$F(x_t) = F(x_0) + \int_0^t \operatorname{grad} F(x_s)\, dx_s + \frac{1}{2}\int_0^t \Delta F(x_s)\, ds =$$

$$= F(x) + \underset{n\to\infty}{\text{l. i. m.}} \sum_{k=0}^{[nt]} \operatorname{grad} F(x_{k/n})\, x_{(k+1)/n}^{k/n} + \frac{1}{2}\int_0^t \Delta F(x_s)\, ds.$$

Multiplying this expression by $\widetilde{\varphi}_t^0$ and taking expectation, after transformations we obtain

$$M_x \widetilde{\varphi}_t^0 F(x_t) = \frac{1}{2}\int_0^t M_x \widetilde{\varphi}_s^0 \Delta F(x_s)\, ds.$$

For an eigenfunction F of the Laplace operator we have $\frac{1}{2}\Delta F = \lambda F$, and $f(t) = M_x \widetilde{\varphi}_{tF}^0(x_t)$ satisfies the Volterra integral equation.

$$f(t) = \lambda \int_0^t f(s)\, ds.$$

It is clear that $f(t) \equiv 0$. For the function F, which is a linear combination of eigenfunctions, (7) is also fulfilled. But any bounded Borel function F can be approximated by linear combinations F_n of the eigenfunctions e^{izx} so that

$$F(x_t) = \underset{n\to\infty}{\text{l. i. m.}}\, F_n(x_t).$$

Therefore equality (7) is fulfilled for all bounded Borel functions, and the functional φ_t^s a.s. (P_x) is equal to zero. The theorem is proved.

Remark 1. Theorem 1 can be generalized to a broad class of processes described by stochastic differential equations

$$dx_t^i = \sum_{j=1}^r \sigma_j^i(x_t)\, d\xi_t^j + m^i(x_t)\, dt, \qquad i = 1, 2, \dots, N.$$

In this case formula (2) is replaced by

$$\varphi_t^s = \int_s^t \sum_{j=1}^r f_j(x_u)\, d\xi_u^j.$$

Remark 2. For **multiplicative** functionals such that $M_x \varphi_t^s = 1$, an analogous theorem holds; namely, every such functional is represented in the form

$$\varphi_t^s = \exp\left\{ \int_s^t f(x_u)\, dx_u - \frac{1}{2} \int_s^t f^2(x_u)\, du \right\}.$$

Remark 3. Functionals having a finite mathematical expectation $M_x \varphi_t^s$ apparently are represented in the form

$$\varphi_t^s = g(x_s) - g(x_t) + \int_s^t f(x_u)\, dx_u,$$

which has so far been proved only for the case when this mathematical expectation does not grow too rapidly as a function of x.

The author has learned that analogous results, independently and by another method, were obtained by A. V. Skorokhod in Kiev.

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Note: Figure translations are in progress. See original paper for figures.

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