MATHEMATICS

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Abstract

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MATHEMATICS

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STRUCTURE OF THE FUNCTION SPACE OF COMPLEX CANONICAL EQUATIONS WITH PERIODIC COEFFICIENTS

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§0. Consider the vector equation

$$J\frac{dx}{dt} = H(t)x, \tag{1}$$

where $H(t)$ is an $n \times n$ complex matrix-function, $H(t) = H(t)^*$, with elements Lebesgue integrable on $(0, 1)$; $H(t+1) = H(t)$ almost everywhere; J is an $n \times n$ constant complex matrix; $J = -J^*$; $\det J \neq 0$.

Denote by L the set of equations (1) for a fixed matrix J, or, equivalently, the set of all matrix-functions $H(t)$ of the above form. After introducing the metric

$$\|H(t)\| = \int_0^1 |H(t)|\, dt$$

the set L becomes a real Banach space*.

The solution matrix (matricant) $X(t)$ of equation (1), normalized by the condition $X(0) = I_n$ (I_n is the $n \times n$ identity matrix), belongs for every t to the group $\mathfrak{G}$ of J-unitary matrices, i.e. satisfies the equation $X^*JX = J$. The eigenvalues of a matrix $X \in \mathfrak{G}$ lying on the unit circle are subdivided, according to the rule of M. G. Krein ([1]), into eigenvalues of the first and second kind. We shall also regard eigenvalues lying inside the unit circle as eigenvalues of the first kind and those outside it as eigenvalues of the second kind. Let p be the number of positive and q the number of negative eigenvalues of the matrix $\frac{1}{i}J$, $p + q = n$. Every matrix $X \in \mathfrak{G}$ has p eigenvalues of the first kind and q of the second kind. By the spectrum ξ of equation (1) we shall mean the totality of eigenvalues of the monodromy matrix $X(1)$ of equation (1), with their kind taken into account. Denote by Σ the set of all spectra of equations (1), $H(t) \in L$. After introducing the natural topology, Σ becomes a linearly connected, locally connected topological space.

Associating with a matrix-function $H(t) \in L$ the monodromy matrix $X(1)$, and with a matrix $X \in \mathfrak{G}$ its spectrum, we obtain mappings of L onto $\mathfrak{G}$ and of $\mathfrak{G}$ onto Σ.

Here we solve the following problem: an arbitrary domain $\widehat{\mathfrak{M}}$ (a linearly connected open set) in Σ is given. It is required to determine into how many domains its complete preimage $\mathfrak{M}$ in L decomposes, and by what properties of the solutions these domains differ. As a similarly specified set $\widehat{\mathfrak{M}}$ one may take the set of all strongly stable or strongly unstable equations**. When specifying properties a), b), it is possible to assume that there is a definite number of solutions of the form

$$x(t) = e^{i\mu t}u(t), \quad u(t+1) = u(t), \quad \mu_1 < \mu < \mu_2$$

with given μ_1, μ_2, or, in addition,

* $|H(t)|$ denotes, for example, the sum of the moduli of the elements of the matrix $H(t)$.

** Equation (1) is called strongly stable (strongly unstable) if all its solutions are bounded as $t \to +\infty$

(among the solutions there are unbounded ones), and this property is preserved under all small changes of th

Moreover, a prescribed number of unbounded solutions, whose characteristic exponents lie within specified prescribed limits, generally different for each solution, etc.

In the case when the matrices $J, H(t)$ are real, $p = q$; $n = 2p$, and $\mathfrak{M}$ is the set of all strongly stable equations, a similar problem was solved earlier by I. M. Gel' fand and V. B. Lidskii ([2]). In the general formulation this problem was solved in the real case for $n = 2$ in ([3],[4]) and for any $n \geq 2$ in ([5]). The complex case, as follows from what is set forth below, differs essentially from the real one.

2°. **Arguments on $\mathfrak{G}$ and on Σ.** Without loss of generality, we shall assume that the matrix J in (1) has the form

$$J = \begin{pmatrix} iI_p & 0 \\ 0 & -iI_q \end{pmatrix}, \tag{2}$$

where I_m is the $m \times m$ identity matrix.

Definition 1. By arguments on the group $\mathfrak{G}$ we shall mean a pair of arbitrary real multivalued functions $\operatorname{Arg}^{(\pm)} X$ satisfying the following conditions: 1)

$\operatorname{Arg}^{(\pm)} X$ are defined for every matrix $X \in \mathfrak{G}$; 2) if $(\operatorname{Arg}^{(\pm)} X)_0$ is one of the values of $\operatorname{Arg}^{(\pm)} X$, then the remaining values are

$$(\operatorname{Arg}^{(\pm)} X)_m = (\operatorname{Arg}^{(\pm)} X)_0 + 2m\pi, \quad m = \dots, -2, -1, 0, 1, 2, \dots;$$

3) each of the branches $(\operatorname{Arg}^{(\pm)} X)_m$ is a continuous function of X; 4) there exist closed curves* $U^{(+)}(t) \in \mathfrak{G}$, $0 \le t \le 1$, such that the increments

$$\Delta \operatorname{Arg}^{(+)} U^{(+)}(t) = \sum_{j=1}^{p} \Delta \operatorname{Arg} \rho_j^{(+)}(t) = 2\pi,$$

$$\Delta \operatorname{Arg}^{(-)} U^{(-)}(t) = \sum_{i=1}^{q} \Delta \operatorname{Arg} \rho_i^{(-)}(t) = 2\pi,$$

where $\rho_j^{(\pm)}(t)$ are, determined by continuity, the eigenvalues of the first $(\rho_j^{(+)}(t))$ and of the second $(\rho_j^{(-)}(t))$ kind of the matrices $U^{(+)}(t)$ and $U^{(-)}(t)$, respectively**.

Obviously, the arguments on $\mathfrak{G}$ will be

$$\operatorname{Arg}^{(\pm)} X = \sum_j \operatorname{Arg} \rho_j^{\pm},$$

where ρ_j^{+} are the eigenvalues of the first kind and $\rho_j^{(-)}$ those of the second kind of the matrix X. It can be shown that arguments on $\mathfrak{G}$ will also be

$$\operatorname{Arg}_0^{(+)} X = \operatorname{Arg} \det U, \qquad \operatorname{Arg}_0^{(-)} X = \operatorname{Arg} \det V,$$

where U, V are determined by decomposing the matrix

$$X = \begin{pmatrix} U & W_1 \\ W_2 & V \end{pmatrix}$$

into four parts; U, V, W_1, W_2 are respectively $p \times p$, $q \times q$, $p \times q$, $q \times p$ matrices. Also arguments are

$$\operatorname{Arg}_{M,N}^{(\pm)} X = \operatorname{Arg}_0^{(\pm)}(MXN),$$

where $M \in \mathfrak{G}$, $N \in \mathfrak{G}$ are certain fixed matrices.

Theorem 1. Let $X(t) \in \mathfrak{G}$ be an arbitrary closed curve, $0 \le t \le 1$. The integers m_+, m_- in the formula

$$\Delta \operatorname{Arg}^{(\pm)} X(t) = 2m_\pm \pi$$

do not depend on the choice of arguments on $\mathfrak{G}$. In order that the curve $X(t)$ can be contracted to a point, it is necessary and sufficient that $m_+ = m_- = 0$.

We shall call the integers m_+, m_- the indices of the closed curve $X(t)$. Introducing in the set

$$\Phi = \{(m_+, m_-)\}$$

of ordered pairs of integers m_+, m_- the operation of componentwise addition, we turn Φ into an Abelian group which, as follows, for example, from Theorem 1, will be isomorphic to the fundamental group ([7]) of the group $\mathfrak{G}$.

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* Here and below, when speaking of a curve in $\mathfrak{G}$, we mean a continuous mapping of the interval $[0, 1]$ into $\mathfrak{G}$.

** The eigenvalues $\rho_j^{(+)}$, as well as $\rho_j^{(-)}$, of the matrix $X \in \mathfrak{G}$, with account taken of kind, are continuous functions of X. This is proved in the same way as in ([5]), Theorem 2.1. $\Delta \operatorname{Arg} \rho(t)$ denotes the increment of the argument of the complex number $\rho(t)$.

Definition 2. Let $\rho_j^{(+)}$ be eigenvalues of the first kind and $\rho_j^{(-)}$ eigenvalues of the second kind of the spectrum $\zeta \in \Sigma$. We shall call the functions

$$\operatorname{Arg}^{(\pm)} \zeta = \sum_j \operatorname{Arg} \rho_j^{(\pm)}$$

the **arguments** on Σ.

The indices m_+, m_- of a closed curve $\zeta(t) \in \Sigma$ are defined by the relations

$$\Delta \operatorname{Arg}^{(\pm)} \zeta(t) = 2\pi m_\pm.$$

3°. Transition from the space Σ to $\mathfrak{G}$. Studying the properties of the mapping $\zeta = \zeta(X)$ in the same way as was done for the real case in ([5]), § 2 and in ([6]), one can obtain the theorem:

Theorem 2. *Under the mapping $\zeta = \zeta(X)$, the full inverse image $\widetilde{\mathfrak{M}} \subset \mathfrak{G}$ of a domain $\widehat{\mathfrak{M}} \subset \Sigma$ is a domain.*

Let us note that the assertion of the theorem will be false if by the spectrum ζ one understands the set of eigenvalues without taking their kind into account.

Passing through a fixed point of some domain $\widetilde{\mathfrak{M}} \subset \mathfrak{G}$ all possible closed curves lying in $\widetilde{\mathfrak{M}}$, consider the set $\Phi_{\widetilde{\mathfrak{M}}} = \{(m_+, m_-)\}$ of their indices. $\Phi_{\widetilde{\mathfrak{M}}}$ will be a subgroup of the group Φ. In an analogous way the group $\Phi_{\widehat{\mathfrak{M}}} \subset \Phi$ of a domain $\widehat{\mathfrak{M}} \subset \Sigma$ is defined.

Theorem 3. *If $\widetilde{\mathfrak{M}} \subset \mathfrak{G}$ is the full inverse image of a domain $\widehat{\mathfrak{M}} \subset \Sigma$, then*

$$\Phi_{\widehat{\mathfrak{M}}} = \Phi_{\widetilde{\mathfrak{M}}}.$$

For a given domain $\widehat{\mathfrak{M}} \subset \Sigma$ the group $\Phi_{\widehat{\mathfrak{M}}}$ is usually easily determined. By Theorem 3 we thereby determine $\Phi_{\widetilde{\mathfrak{M}}}$. It is known ([8]) that an arbitrary subgroup

Φ_0 of the group Φ has either rank equal to one and a basis $(\chi p_1, \chi q_1)$, where p_1, q_1 are relatively prime integers, $\chi > 0$, or rank equal to two and a basis $(\chi_1 p_1, \chi_1 q_1)$, $(\chi_2 p_2, \chi_2 q_2)$, where $p_1 q_2 - p_2 q_1 = 1$, $\chi_1 > 0$, $\chi_2 > 0$, χ_2 is divisible by χ_1 (χ_1, χ_2 are the torsion coefficients of the group Φ_0).

4°. Transition from $\mathfrak{G}$ to L. Let $\widetilde{\mathfrak{M}}$ be a domain in $\mathfrak{G}$; $\mathfrak{M}$ its full inverse image in L; $H(t) \in \mathfrak{M}$; $X(t)$ the matricant of the corresponding equation (1); $X(1) \in \widetilde{\mathfrak{M}}$. Choose an arbitrary fixed point $Z_0 \in \widetilde{\mathfrak{M}}$ and connect Z_0 with some fixed curve $Z(t) \in \mathfrak{M}$ with the matrix I_n; $Z(0) = Z_0$; $Z(1) = I_n$. Connect arbitrarily $X(1)$ by a curve $Y(t) \in \mathfrak{M}$ with the matrix Z_0; $Y(0) = X(1)$; $Y(1) = Z_0$. The closed curve $X'(t)$, composed successively of the curves $X(t), Y(t), Z(t)$, will be called the augmented matricant.

It can be shown that to each coset class $\nu \in \Phi/\Phi_{\widetilde{\mathfrak{M}}}$ there corresponds a certain domain $\mathfrak{M}_\nu \subset \mathfrak{M}$ such that, for $H(t) \in \mathfrak{M}_\nu$, the indices m_+, m_- of the corresponding augmented matricant satisfy the condition $(m_+, m_-) \in \nu$. Moreover, $\mathfrak{M}_{\nu_1}, \mathfrak{M}_{\nu_2}$, corresponding to different ν_1, ν_2, do not intersect and $\mathfrak{M} = \bigcup_\nu \mathfrak{M}_\nu$. Hence it follows:

Theorem 4. 1) *Let $\Phi_{\widetilde{\mathfrak{M}}} = \{(0,0)\}$ be the trivial subgroup. The set $\mathfrak{M}$ decomposes into a countable number of nonintersecting domains $\mathfrak{M}_{k_1 k_2}$, $k_1, k_2 = 0, \pm 1, \pm 2, \ldots$. The domain $\mathfrak{M}_{k_1 k_2}$ consists of all $H(t) \in \mathfrak{M}$ such that the corresponding augmented matricants $X'(t)$ have indices $m_+ = k_1$, $m_- = k_2$.* 2) *Let $\Phi_{\widetilde{\mathfrak{M}}}$ be a subgroup of rank one, $(\chi p_1, \chi q_1)$ its basis, and p_2, q_2 integers such that $p_1 q_2 - p_2 q_1 = 1$. The set $\mathfrak{M}$ decomposes into a countable number of nonintersecting domains $\mathfrak{M}_{k,r}$, $k = 0, \pm 1, \pm 2, \ldots$, $r = 0, 1, \ldots, \chi$. The set $\mathfrak{M}_{k,r}$ consists of all $H(t) \in \mathfrak{M}$ for which the corresponding augmented matricants $X'(t)$ have indices m_+, m_- satisfying the relations*

$$\begin{vmatrix} m_+ & m_- \\ p_1 & q_1 \end{vmatrix} = k, \qquad \begin{vmatrix} m_+ & m_- \\ p_2 & q_2 \end{vmatrix} = r \pmod{\chi}. \tag{3}$$

3) *Let $\Phi_{\widetilde{\mathfrak{M}}}$ be a subgroup of rank two; $(\chi_1 p_1, \chi_1 q_1)$, $(\chi_2 p_2, \chi_2 q_2)$ its basis; $p_1 q_2 - p_2 q_1 = 1$. The set $\mathfrak{M}$ decomposes into $\chi_1 \chi_2$ domains $\mathfrak{M}_{r_1, r_2}$, $r_1 = 0, 1, \ldots, \chi_1$, $r_2 = 0, 1, \ldots, \chi_2$. The domain $\mathfrak{M}_{r_1, r_2}$ consists of all $H(t) \in \mathfrak{M}$,*

for which the corresponding completed matrixants have index m_+, m_-, satisfying the relations

$$\begin{vmatrix} m_- & m_+ \\ p_1 & q_1 \end{vmatrix} \equiv r_2 \pmod{\chi_2}, \qquad \begin{vmatrix} m_+ & m_- \\ p_2 & q_2 \end{vmatrix} \equiv r_1 \pmod{\chi_1}.$$

Theorems 3 and 4 solve the problem formulated in 1°.

5°. **The structure of the regions of stability.** From the results of M. G. Krein, I. M. Gel' fand, and V. B. Lidskii ([1,2]) it follows that equation (1) is strongly stable if and only if its spectrum ζ lies on the unit circle and in the spectrum there are no coincident eigenvalues of different kind. It is easy to show that the corresponding set of spectra $\hat{\mathfrak{D}} \subset \Sigma$ decomposes into N_{pq} regions $\hat{\mathfrak{D}}^{(\mu)}$, $\mu = \mu_1, \dots, \mu_{Npq}$, where

$$N_{pq} = \sum_{r=1}^{\min(p,q)} \left(\frac{1}{r} C_{p-1}^{r-1} C_{q-1}^{r-1} + \frac{r-1}{r} \sigma_{p,r} \sigma_{q,r} \right);$$

$\sigma_{m,r} = 0$, if $m \not\equiv 0 \pmod r$; $\sigma_{m,r} = 1$, if $m \equiv 0 \pmod r$.

Since the matrix I_n lies on the boundary of $\hat{\mathfrak{D}}^{(\mu)}$, by choosing the matrix Z_0 in a suitable manner we obtain that by the completed matrixant $X'(t)$ one may understand a curve composed successively of the matrixant $X(t)$ and a curve $Y(t)$ such that $Y(0) = X(1)$, $Y(1) = I_n$, $\zeta[Y(t)] \in \hat{\mathfrak{D}}^{(\mu)}$ for $0 \le t < 1$, and for t sufficiently close to 1, the spectrum $\zeta[Y(t)]$ lies on the arc $\rho = e^{i\varphi}$, $0 < \varphi < \varepsilon_0 < \pi$.

The group $\Phi_{\hat{\mathfrak{D}}^{(\mu)}}$ is computed without difficulty: $\Phi_{\hat{\mathfrak{D}}^{(\mu)}}$ does not depend on μ, has rank one and basis (p, q). From Theorem 4 it follows:

Theorem 5. Let $\chi > 0$ be the greatest common divisor of the numbers p, q, so that $p = \chi p_1$, $q = \chi q_1$, where p_1 and q_1 are relatively prime. Let p_2, q_2 be integers such that $p_1 q_2 - p_2 q_1 = 1$. The set $\mathfrak{D}$ of all strongly stable $H(t) \in L$ decomposes into a countable number of regions $\mathfrak{D}_{k,r}^{(\mu)}$ ("regions of stability"), each of which is characterized by one of the N_{pq} possible types μ of the spectrum and by the integers $k = 0, \pm 1, \pm 2, \dots$, $r = 0, 1, \dots, \chi$, which are determined by the relations (3), where m_+, m_- are the indices of the corresponding completed matrixants.

6°. **The structure of the regions of instability.** For strong instability it is obviously sufficient that at least one point of the spectrum lie outside the unit circle. It can be shown that this condition is also necessary. Denote by $\mathfrak{H}$ the set of all strongly unstable $H(t)$; $\tilde{\mathfrak{H}}, \hat{\mathfrak{H}}$ are its projections in $\mathfrak{S}$ and Σ. By the completed matrixant $X'(t)$ one may now understand a curve composed successively of the matrixant $X(t)$ and a curve $Y(t)$, $0 \le t \le 1$, $Y(0) = X(1)$, $Y(1) = I_n$, such that at least one point of the spectrum $\zeta[Y(t)]$, for $0 \le t < 1$, lies outside the unit circle.

Theorem 6. If $p > 1$ or $q > 1$, the set $\mathfrak{H}$ is a region. If $p = q = 1$, the set $\mathfrak{H}$ decomposes into a countable number of regions $\mathfrak{H}_k$, $k = 0, \pm 1, \pm 2, \dots$. The region $\mathfrak{H}_k$ consists of all $H(t) \in L$ for which the corresponding completed matrixants satisfy the condition

$$\Delta \operatorname{Arg}^{(+)} X'(t) - \Delta \operatorname{Arg}^{(-)} X'(t) = 2\pi k.$$

In an analogous manner one can without difficulty determine the structure of any of the sets listed in item 1°.

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Note: Figure translations are in progress. See original paper for figures.