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Abstract

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ON HUREWICZ' S THEOREM

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In § 1 the sum theorem is extended to point-finite open covers*, and Hurewicz' s theorem on lowering dimension under mappings is proved in Morita' s form ([1])
$\dim X \leq \operatorname{Ind} Y + \dim f$** under the assumption of weak paracompactness*** of the image. In § 2 the notions of a fully paracompact space and of a fully paracompact decomposition are introduced and studied. For metrizable spaces the first notion coincides with the notion of strong metrizability—the existence of a base decomposable into the sum of a countable number of star-finite**** covers (apparently introduced by Morita). An example is given of a strongly metrizable space which is not decomposable into the sum of a countable number of closed strongly paracompact***** spaces. In § 3 the same Hurewicz theorem is proved in the form of Yu. Smirnov ([2])
$\dim X \leq \operatorname{ind} Y + \dim f$ for fully paracompact decompositions, in particular for fully paracompact spaces. For these spaces the inequality $\dim X \leq \operatorname{ind} X$ follows from this.

§ 1. **Theorem 1.** *If a normal space R has a point-finite open cover ω such that* $\dim U \leq n$ *whenever $U \in \omega$, then* $\dim R \leq n$.

Lemma 1. *For a point-finite open cover ω, for each $k = 1, 2, \ldots$ the set T_k, consisting of all those points each of which belongs to no more than k elements of the cover ω, is closed; for every neighborhood O of the set T_{k-1}, the difference $T_k \setminus O$ is the body****** of a closed discrete******* system inscribed in ω.*

For the proof of the theorem we prove by induction that $\dim T_k \leq n$. Indeed, if $\Phi \subseteq T_k \setminus T_{k-1}$ and Φ is closed, then, by the lemma, $\dim \Phi \leq n$. Hence, by Dowker' s lemma ([3]), we have $\dim T_k \leq n$.

Corollary. *If the space R is normal and weakly paracompact, then* $\operatorname{loc}\dim R = \dim R$********.

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* A cover is called **point-finite** if each point is contained in only finitely many elements of this cover.

** The dimension dim is the dimension defined by means of covers; Ind is the large inductive dimension, ind the small inductive dimension;

$\dim f = \sup_{y \in Y} \dim f^{-1} y$.

*** A space is **weakly paracompact** if every one of its open covers has an inscribed point-finite open cover.

**** A cover is called **star-finite** if each of its elements intersects only finitely many other elements.

***** A space is **strongly paracompact** if every one of its open covers has an inscribed open star-finite cover.

****** The **body of a system** is the union of all elements of this system.

******* A system of sets A_λ is **discrete** if every point $x \in R$ has a neighborhood intersecting no more than one set A_λ.

******** The **local dimension** $\operatorname{loc}\dim R$ of the space R is the least of those numbers n such that every point $x \in R$ has a neighborhood Ox such that $\dim[Ox] \le n$ (see ([3])).

Theorem 2. Let g be a closed* mapping of a normal space X onto a weakly paracompact normal space Y; then

$$\dim X \le \operatorname{Ind} Y + \dim g.$$

Proof. Suppose that, in the case $\operatorname{Ind} Y' \le m-1$, the theorem is true, and let $\operatorname{Ind} Y = m$. Let Φ be closed in X, and let f be a mapping of the set Φ into the sphere S^n, $n = m + \dim g$. Since $\dim g^{-1}(y) \le n$, f can be extended to $\Phi \cup g^{-1} y$ and, hence, to some neighborhood V_y of it. The sets

$$O_y = Y \setminus g(X \setminus V_y)$$

form an open cover of the space Y. We inscribe in it a locally finite cover ω. Let $S_k = g^{-1}(T_k)$. By induction one can prove that there exist open sets Γ_k satisfying the following conditions: a)

$$\bigcup_{k \le l} S_k \subseteq \bigcup_{k \le l} \Gamma_k$$

for every l; b) $\dim \operatorname{Fr} \Gamma_k \le n-1$; c) $G_k = g(\Gamma_k)$ is open in Y; d) $g^{-1} G_k = \Gamma_k$; e) the mapping f is extended to $\Phi \cup [\Gamma_k]$. By Hurewicz' s lemma, generalized by Yu. Smirnov in ([2]), the mapping f is extended to all of X. Hence $\dim X \le n$, as was required to prove.

2. We shall say that a cover β is weakly inscribed in a cover α if one can choose from β a subcover inscribed in α.** We shall call a regular space fully paracompact if, in every one of its open covers, one can weakly inscribe an open cover decomposing into the sum of a countable number of star-finite covers.

Theorem 3. Every fully paracompact space is paracompact; every strongly paracompact space is fully paracompact.

For the proof of the first assertion, in view of a well-known theorem of Michael ([4]), only the following is needed.

Lemma 2. In every open cover of a fully paracompact space one can inscribe an open cover decomposing into the sum of a countable number of discrete subsystems.

The lemma follows from the fact that every open star-finite cover ω_i decomposes into the sum of countable or finite subsystems $\omega_{i\lambda}$, whose bodies are pairwise disjoint and are open-closed sets ([5]). Taking one element from each subsystem $\omega_{i\lambda}$, we obtain a discrete system. Therefore the sum of the star-finite covers ω_i also decomposes into the sum of a countable number of discrete subsystems.

We shall call a regular space **strongly metrizable** if it has a base decomposing into the sum of a countable number of star-finite covers.***

Lemma 3. A metrizable space is strongly metrizable if and only if it is fully paracompact.

It is easy to see that the product of a countable number of strongly metrizable spaces is strongly metrizable and that subspaces of a strongly metrizable space are also strongly metrizable. For strongly paracompact spaces these properties do not hold ([6]).

Example 1. A strongly metrizable space S that is not the sum of a countable number of closed strongly paracompact sets.

Let $\prod I_k$ be the topological product of a countable number of (open) intervals I_k, and let B^τ be the Baire space of uncountable weight ([5])τ. Then

$$S = B^\tau \times \prod I_k.$$

One can show that every neighborhood of the space S contains a closed subset of S that is not strongly paracompact. Hence every closed strongly paracompact set is nowhere dense in S. From this everything follows by the completeness of S and Baire' s theorem.

* A continuous mapping is called **closed** if the image of every closed set is closed.

** Every base of open sets is weakly inscribed in every open cover of the given space.

*** By the metrization theorem of Nagata–Smirnov ([7]), every strongly metrizable space is metrizable.

Theorem 4. *Every set of type F_σ in a completely paracompact space is completely paracompact.*

Example 2. A closed mapping that maps a strongly paracompact metrizable space into a space that is not completely paracompact.

Take an uncountable number of discretely arranged segments of length 1 and identify the initial points of these segments into one point, so as to obtain a "nonmetrizable hedgehog."

Note that, by mapping a "metrizable hedgehog" to one point, we obtain a closed mapping of a metrizable, not strongly paracompact space onto a compactum.

By a **decomposition** of a space R we shall mean a system of pairwise disjoint closed sets whose sum is equal to R.* A covering γ will be called a **covering of the decomposition** β if β is inscribed in γ. A decomposition β of a space R will be called **completely paracompact** if in every open covering of this decomposition one can weakly inscribe an open covering of the space R which decomposes into the sum of a countable number of star-finite coverings of the space R.

Lemma 4. *Let g be a closed mapping of a space X onto a space Y; if one of these spaces is completely paracompact, then the decomposition generated by the mapping g is also completely paracompact.*

In the case of complete paracompactness of the space X, the closedness of the mapping is not needed.

Example 3. A closed mapping of a space that is not completely paracompact onto a space that is not completely paracompact, generating a completely paracompact decomposition.

Let X be the sum of an uncountable number of discretely arranged "metrizable hedgehogs." In each of them choose one segment $A_\lambda O_\lambda$, where O_λ is the center of the hedgehog. Map each segment $A_\lambda O_\lambda$ isometrically onto the corresponding segment $A'_\lambda O'$ of the "nonmetrizable hedgehog" Y. Map all remaining points of the space X to the center O' of the hedgehog Y.

p. 3. **Theorem 5.** *Let in a normal space R, for each neighborhood O of each element F of a decomposition β, there exist a neighborhood V such that $F \subset V \subset O$ and such that $\dim \operatorname{Fr} V \leq n-1$; then, if β is completely paracompact and if $\dim F \leq n$ for every $F \in \beta$, then also $\dim R \leq n$.*

Lemma 5. *Let an open covering γ weakly contain an open covering decomposing into the sum of a countable number of star-finite coverings; then in γ one can inscribe an open covering, decomposing into the sum of a countable number of discrete systems, such that the boundary of each of its elements will lie in the boundary of some element of the covering γ.*

Proof of the theorem. Let β be a completely paracompact decomposition, and let Φ be a closed set of the space R. Let f be a mapping of the set Φ into the sphere S^n. Just as before, f can be extended to some neighborhood O_F of each set $F \in \beta$. We may assume that $\dim \operatorname{Fr} O_F \leq n-1$. By Lemma 5 one can find such open sets Γ_i that $\dim \operatorname{Fr} \Gamma_i \leq n-1$ and such that f is extendable to $\Phi \cup [\Gamma_i]$ for every $i = 1, 2, \ldots$. Hence, f is extendable to all of R, and $\dim R \leq n$. The theorem is proved.

Theorem 6. *Let g be a closed mapping of a normal space X onto a normal space Y; if the mapping g generates a completely paracompact decomposition of the space X, then $\dim X \leq \operatorname{ind} Y + \dim g$.*

Lemma 6. *Let $\beta = \{F\}$ be a decomposition of a space R, and let β' be such a subsystem of it that the sum*

$$A = \bigcup_{F \in \beta'} F$$

is closed; then complete para-

* Every continuous mapping g generates a decomposition of the preimage X into the full preimages $g^{-1}(y)$ of points $y \in Y$.

compactness of the decomposition β entails the complete paracompactness of the decomposition β' on A.

Proof of the theorem. Suppose that in the case $\operatorname{ind} Y' < k$ the theorem is true, and let $\operatorname{ind} Y = k$. Apply Theorem 5. For an arbitrary neighborhood O of the inverse image $g^{-1}(y)$, by virtue of the closedness of the mapping g there exists a neighborhood Oy of the point y such that $g^{-1}(Oy) \subseteq O$ and $\operatorname{ind} \operatorname{Fr} Oy \leq k - 1$. Lemma 6 together with the induction hypothesis leads to the inequality
$\dim g^{-1}(\operatorname{Fr} Oy) \leq \dim g + k - 1$, and hence also to the inequality
$\dim \operatorname{Fr} g^{-1}(Oy) \leq \dim g + k - 1$, which was required to be proved.

Corollary 1. If g is a closed mapping of a normal space X onto a normal space Y, then the inequality
$\dim X \leq \dim g + \operatorname{ind} Y$ holds in the following two cases: 1) either X or Y is the sum of a countable number of closed completely paracompact sets; 2) or X or Y is the sum of completely paracompact sets forming a locally finite system, all of which, except possibly one, are closed.

This is derived from Lemma 4 by Dowker' s method from ([3]).

Corollary 2. For every normal space R satisfying one of the conditions of the preceding corollary, in particular for a completely paracompact space, one always has
$\dim R \leq \operatorname{ind} R$.

Hence, from the well-known theorem of Katetov—Morita ([8]), we have:

Corollary 3. For every metrizable space R that decomposes into the sum of a countable number of completely paracompact (strongly metrizable) closed sets, the fundamental dimensions are equal:
$\dim R = \operatorname{ind} R = \operatorname{Ind} R$.

Theorem 7. If in a completely paracompact space R the sum theorem holds for the dimension Ind, then
$\operatorname{Ind} R = \operatorname{ind} R$.

The proof is based on the following lemma:

Lemma 7. Let R be a completely paracompact space of dimension $\operatorname{ind} R \leq n$; then for every closed set A and every neighborhood OA of it there exists a neighborhood UA such that $UA \subseteq OA$ and such that the boundary $\operatorname{Fr} UA$ decomposes into the sum of a countable number of closed sets C_i of dimension $\operatorname{ind} C_i \leq n-1$.

Remark. The corollary of Theorem 1 and Theorem 2 are true under the following assumptions: 1) the space $R(Y)$ is the sum of a countable number of closed weakly paracompact sets; 2) $R(Y)$ is the body of a locally finite system of weakly paracompact sets, all of which are closed except, possibly, one.

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Note: Figure translations are in progress. See original paper for figures.