MATHEMATICS

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. KUZ' MINOV

ON CONTINUA V^n

(Presented by Academician P. S. Aleksandrov on 23 February 1961)

P. S. Aleksandrov proved (^{1}) that every n-dimensional bicompactum contains an n-dimensional Cantor manifold. He also proposed the following strengthening of the notion of a Cantor manifold: an n-dimensional bicompactum X is called a **continuum** V^n if, for every pair of disjoint open sets H and G in X, there exists a covering ω of the space X such that no partition C separating the sets H and G can be ω-mapped into an $(n-2)$-dimensional polyhedron. In (^{2}) P. S. Aleksandrov proved that every n-dimensional compactum contains a continuum V^n, and posed the problem of determining whether every n-dimensional bicompactum contains a continuum V^n. In the present note a positive solution of this problem is given.

Let X be a bicompactum, and A its closed subset. By $H^q(X, A)$ we denote the q-dimensional Aleksandrov-Čech cohomology group of the pair (X, A) with coefficients in an arbitrary group G, which we shall assume fixed. The cohomological dimension of the bicompactum X over the coefficient domain G will be denoted by $\operatorname{cd}_G X$.

Definition 1. A bicompactum X at a point $a \in X$ forms a q-dimensional obstacle if there exists a neighborhood V of the point a such that, for every neighborhood U of the point a contained in V, the homomorphism

$$i^* : H^q(X, X \setminus U) \to H^q(X, X \setminus V),$$

induced by the inclusion map

$$(X, X \setminus V) \to (X, X \setminus U),$$

is nontrivial.

Definition 2. A point a of a bicompactum X, $\operatorname{cd} X = q$, is called **basic** if, for every sufficiently small neighborhood W of the point a, the homomorphism

$$i^* : H^{q-1}(\overline{W}) \to H^{q-1}(\dot{W})$$

is not a mapping onto the whole group $H^{q-1}(\dot{W})$ (by $\overline{W}$ and $\dot{W}$ are denoted, respectively, the closure and the boundary of the set W).

Definitions equivalent to these were given by P. S. Aleksandrov for compacta lying in Euclidean space.

Theorem 1. *Let X be a bicompactum, $\operatorname{cd} X = n$; then there exist in X closed sets Y and A and a set H open in Y such that the following conditions are satisfied:*

1°. *H consists of basic points of the bicompactum Y.*

2°. *$\overline{H} = Y$; H contains a bicompactum whose cohomological dimension is equal to n.*

3°. *At every point $a \in H$ the bicompactum X forms an n-dimensional obstacle.*

4°. *For any sets G_1 and G_2 open in Y there exists a covering ω of the space Y such that, for every partition C separating the sets G_1 and G_2, the homomorphism*

$$\pi^*_\omega : H^{n-1}(N_{\omega|C}, N_{\omega|C\cap A}) \to H^{n-1}(C, C \cap A)$$

is nontrivial.

Here $N_{\omega|C}$ denotes the nerve of the covering induced by the covering ω on the set C; the notation $N_{\omega|C\cap A}$ has an analogous meaning; π_ω denotes-

defines a homomorphism from the projection of the nerve cohomology group of the pair $(C, C \cap A)$ into the inverse limit group of the direct spectrum defining the cohomology group $H^{n-1}(C, C \cap A)$.

Proof. Since $\operatorname{cd} X = n$, there exists a closed subset B of the bicompactum X for which $H^n(X, B) \neq 0$. Let $e \in H^n(X, B)$ and $e \neq 0$.

Using the continuity of the spectral cohomology groups, we find a bicompactum $Y \subset X$ such that the image of the element e under the homomorphism

$$i^* : H^q(X, B) \to H^q(Y, Y \cap B)$$

is nonzero, while for every proper closed subset Y' of the bicompactum Y the image of the element e under the homomorphism

$$i_1 : H^q(X, B) \to H^q(Y_1, Y_1 \cap B)$$

is equal to zero. Let $A = Y \cap B$, $\overline{H} = Y \setminus B$, and $i^* e = e_1$. We shall show that the sets Y, A, and H satisfy all the requirements of the theorem.

Let $a \in H$, and let W be a neighborhood of the point a in the bicompactum Y such that $\overline{W} \cap A = \emptyset$. Consider the addition sequence of the triad $(Y, \overline{W}, Y \setminus W)$. Taking into account that $\overline{W} \cap (Y \setminus W) = \dot{W}$ and $\overline{W} \cap A = \emptyset$, this sequence may be written in the following form:

$$\hat{H}^{n-1}(\overline{W}) + H^{n-1}(Y \setminus \overline{W}, A) \xrightarrow{\psi} H^{n-1}(\dot{W}) \xrightarrow{\Delta}$$

$$\longrightarrow H^n(Y, A) \xrightarrow{\varphi} H^n(\overline{W}) + H^n(Y \setminus \overline{W}, A).$$

Since the image of the element e in the cohomology group of a proper closed subset of the bicompactum Y, under the homomorphism induced by inclusion, is zero, we have $\varphi(e_1) = 0$. From exactness of the sequence there follows the existence of an element $e_2 \in H^{n-1}(\dot{W})$ for which $\Delta e_2 = e_1$. Then the image of the homomorphism ψ, and consequently also of the homomorphism

$$i_2^* : H^{n-1}(\overline{W}) \to H^{n-1}(\dot{W}),$$

does not contain the element e_2; thus assertion 1° is proved.

Suppose that $\overline{H} \neq Y$. Then A contains a set M open in Y. By the excision theorem the mapping

$$i^* : H^n(Y, A) \to H^n(Y \setminus M, A \setminus M)$$

is an isomorphism, which, however, contradicts the "minimality" property of the bicompactum Y. Thus $\overline{H} = Y$.

By exactness of the sequence

$$H^{n-1}(\overline{W}) \xrightarrow{i_2^*} H^{n-1}(\dot{W}) \xrightarrow{\partial} H^n(\overline{W}, \dot{W}),$$

the element ∂e_2 is different from zero, and therefore the group $H^n(\overline{W}, \dot{W})$ is nontrivial.

Thus the set H contains the bicompactum $\overline{W}$, whose cohomological dimension is equal to n, and assertion 2° is proved.

Let $a \in H$, $V = X \setminus A$, and let U be a neighborhood of the point a contained in V. Consider the commutative diagram:

$$\begin{tikzcd}
H^n(X, X \setminus U) + H^n(X, Y) \arrow[r, "\psi"] & H^n(X, (X \setminus U) \cap Y) \arrow[r, "\Delta"] \arrow[d, "i^*"] & 0 \\
& H^n(X, B) \arrow[r, "l^*"] \arrow[d, "l^*"] & H^n(X, A) \arrow[d, "k^*"] \\
& H^n(Y, A) & H^n((X \setminus U) \cap Y, A) \arrow[l, "m^*"']
\end{tikzcd}$$

The upper row of this diagram is a segment of the relative addition cohomology sequence of the triad $(X, X \setminus U, Y)$, the last column is a segment of the cohomology sequence of the triple $(X, (X \setminus U) \cap Y, A)$, and i^*, j^*, k^*, l^*, m^* are homomorphisms of cohomology groups induced by inclusion mappings. Let $e \in H^n(X, B)$ be the element chosen earlier. Then $m^* l^* e = e_1$, $e_1 \neq 0$. Consequently, $l^* e \neq 0$. Further, $k^* l^* e = 0$ by the "minimality" property of the set Y. From exactness of the column and row of the diagram there follows the existence of such elements

$e_3 \in H^n(X, (X \setminus U) \cap Y)$ and $(e_4, e_5) \in H^n(X, X \setminus U) + H^n(X, Y)$, for which $j^* e_3 = l^* e$ and $\psi(e_4, e_5) = e_3$. But

$$\psi(e_4, e_5) = -i_3^* e_4 + i_4^* e_5,$$

where

$$i_3^* : H^n(X, X \setminus U) \to H^n(X, (X \setminus U) \cap Y)$$

and

$$i_4^* : H^n(X, Y) \to H^n(X, (X \setminus U) \cap Y).$$

If $i^* l_3^* e_4 = 0$, then $m^* j^* i_4^* = e_1$. But the homomorphism

$$m^* j^* i_4^* : H^n(X, Y) \to H^n(Y, A)$$

is, obviously, trivial. Consequently, $j^* i_3^* e_4 \neq 0$, and therefore the homomorphism

$$j^* i_3^* : H^n(X, X \setminus U) \to H^n(X, X \setminus V)$$

is nontrivial. Thus assertion 3° is proved.

Let G_1 and G_2 be disjoint open subsets of Y; $F_1 = Y \setminus G_2$ and $F_2 = Y \setminus G_1$; let C be a partition separating the sets G_1 and G_2, and let F_3 and F_4 be closed sets such that $F_3 \cap F_4 = C$ and $G_2 \subseteq Y \setminus F_3$, $G_1 \subseteq Y \setminus F_4$.

Consider the commutative diagram

$$\begin{tikzcd}
H^{n-1}(F_1 \cap F_2, F_1 \cap F_2 \cap A) \arrow[r, "\Delta"] \arrow[d, "i^*"] & H^n(Y, A) \arrow[r, "\psi"] \arrow[d] & H^n(F_1, F_1 \cap A) + H^n(F_2, F_2 \cap A) \arrow[d] \\
H^{n-1}(C, C \cap A) \arrow[r, "\Delta"] & H^n(Y, A) & H^n(F_3, F_3 \cap A) + H^n(F_4, F_4 \cap A).
\end{tikzcd}$$

The rows of this diagram are segments of the additive cohomology sequences of the triads (Y, F_1, F_2) and (Y, F_3, F_4).

Let e_1 be the previously chosen element of the group $H^n(Y, A)$. Then $\psi(e_1) = 0$, and therefore in the group

$$H^{n-1}(F_1 \cap F_2, F_1 \cap F_2 \cap A)$$

there exists an element e_6 such that $\Delta e_6 = e_1$. Choose a cover ω of the space Y such that the element e_6 is contained in the image of the homomorphism

$$\pi_\omega : H^{n-1}(N_{\omega | F_1 \cap F_2}, N_{\omega | F_1 \cap F_2 \cap A}) \to H^{n-1}(F_1 \cap F_2, F_1 \cap F_2 \cap A).$$

If f is an element of the group

$$H^{n-1}(N_{\omega | F_1 \cap F_2}, N_{\omega | F_1 \cap F_2 \cap A})$$

such that $\pi_\omega f = e_6$, and $g = i^* f$, where

$$i : H^{n-1}(N_{\omega | F_1 \cap F_2}, N_{\omega | F_1 \cap F_2 \cap A}) \to H^{n-1}(N_{\omega | C}, N_{\omega | C \cap A})$$

is the mapping induced by the inclusion of the nerves of the covers, then

$$\pi_\omega g = \pi_\omega i^* f = i^* \pi_\omega f = i^* e_6.$$

Since $\Delta i^* e_6 = e_1$, it follows that $i^* e_6 \neq 0$, and hence the homomorphism

$$\pi_\omega : H^{n-1}(N_{\omega|C}, N_{\omega|C\cap A}) \to H^{n-1}(C, C \cap A)$$

is nontrivial. Thus Theorem 1 is completely proved.

Corollary 1. *A bicompactum X has cohomological dimension n if and only if it forms an n-dimensional obstruction at at least one point and at no point forms an obstruction of greater dimension.*

Corollary 2. *Let X be a bicompactum, $\operatorname{cd} X = n$. Then X contains a bicompactum Y, which is the closure of a set H open in it, all points of which are basic points of the bicompactum Y.*

Corollary 3. *The set of points at which an n-dimensional bicompactum forms an n-dimensional obstruction contains an n-dimensional bicompactum.*

Corollaries 1 and 2 are generalizations of known theorems of P. S. Aleksandrov on the homological dimension of compacta.

Theorem 2. *Every n-dimensional bicompactum X contains a continuum V^n.*

Proof. We shall show that, if in Theorem 1 the group Z of integers is taken as the coefficient group G, then the bicompactum Y, whose existence is asserted in Theorem 1, is a continuum V^n. Indeed, in the case under consideration $\operatorname{cd} X = \dim X = n$. For any open subsets G_1 and G_2 of Y, by Theorem 1 there will be found a cover ω such that the mapping

$$\pi_n : H^{n-1}(N_{\omega|C}, N_{\omega|C\cap A}) \to H^{n-1}(C, C \cap A)$$

is nontrivial for every partition C separating the sets G_1 and G_2. Let f be some ω-mapping of the partition C into a polyhedron of dimension $\leq n-2$. Then into the cover $\omega|C$ one can inscribe a cover α of multiplicity $\leq n-1$. For such a cover α the relations

$$H^{n-1}(N_{\alpha|C}, N_{\alpha|C\cap A}) = 0$$

and

$$\pi_\omega = \pi_\alpha \pi_\omega^\alpha$$

hold. Since the homomorphism π_ω is nontrivial, the homomorphism

$$\pi_\alpha : H^{n-1}(N_{\alpha|C}, N_{\alpha|C\cap A}) \to H^{n-1}(C, C \cap A)$$

is also nontrivial.

This contradicts the fact that $H^{n-1}(N_{\alpha|C}, N_{\alpha|C\cap A}) = 0$. Consequently, no ω-maps f of the indicated form exist, and therefore Y is a V^n-continuum.

I take this opportunity to express my gratitude to I. A. Shvedov for the great help he gave me in writing this paper.

Moscow State University
named after M. V. Lomonosov

Received
23 II 1961

References

[1] P. Alexandroff, Proc. Roy. Soc. London, A **189**, 11 (1947).
[2] P. Alexandroff, Monatsh. f. Math., **61**, No. 1, 67 (1957).
[3] P. Aleksandrov, Uspekhi Mat. Nauk, **4**, issue 6 (1949).
[4] E. Cohen, Duke Math. J., **21**, 209 (1954).
[5] N. Steenrod, C. Eilenberg, *Foundations of Algebraic Topology*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.