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Abstract

Full Text

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ON HOMOGENEOUS RIEMANNIAN NON-SYMMETRIC SPACES WITH IRREDUCIBLE GROUP OF ROTATIONS

(Presented by Academician I. G. Petrovskii, 12 VI 1961)

1°. Let $M = \mathfrak{G}/\mathfrak{H}$ (the group \mathfrak{H} is compact) be a nonsymmetric homogeneous Riemannian space with irreducible group of rotations. Our problem is to find all such spaces when \mathfrak{G} is simple and of type A_m (if \mathfrak{G} is nonsimple or noncompact, the spaces sought do not exist). \hat{G} and H denote the Lie algebras of the groups \mathfrak{G} and \mathfrak{H} . Define the action of H on \hat{G} by the formula

$$h(g) = [h, g], \quad h \in H, \quad g \in \hat{G}, \quad (1)$$

where $[h, g]$ is the commutator of the vectors in \hat{G} . Formula (1) gives a representation of the algebra H . The compact group \mathfrak{H} preserves some positive definite quadratic form γ in the space \hat{G} . Therefore

$$\hat{G} = H + B, \quad [H, B] \subset B, \quad \gamma(h, b) = 0, \quad h \in H, \quad b \in B. \quad (2)$$

The space B is identified with the tangent space to M . The infinitesimal transformations of \mathfrak{H} transform B in the following way:

$$h(b) = [h, b], \quad h \in H, \quad b \in B. \quad (3)$$

By the property of the space M , this representation of H on B must be irreducible. It is said that the pair $H \subset \hat{G}$ has a symmetric structure if, in addition to formulas (2), the relation

$$[B, B] \subset H \quad (4)$$

is satisfied.

Pairs of symmetric structure generate symmetric spaces.

2°. In what follows, by \mathfrak{G} we mean SU (the unitary unimodular group), and by \mathfrak{G}^+ the complex unimodular group SL^+ . Then \mathfrak{H} is embedded in \mathfrak{G} , and \mathfrak{H}^+ , the complex form of \mathfrak{H} , is embedded in \mathfrak{G}^+ by means of a certain linear representation φ .

Dynkin diagrams for $A_m, B_m, C_m, D_m, E_6, E_7, E_8, F_4, G_2$

Figure 1: Dynkin diagrams for $A_m, B_m, C_m, D_m, E_6, E_7, E_8, F_4, G_2$

1. If the compact linear group \mathfrak{H} on the real space B is irreducible, then \mathfrak{H}^+ on the complex B^+ either is irreducible, or decomposes into two inequivalent mutually contragredient representations.

We shall call the problem of finding all pairs $H^+ \subset G^+$ (H^+ is the complex envelope of a compact algebra, G^+ is the algebra of complex matrices with trace zero) such that the conclusion of Theorem 1 is satisfied problem A.

2. If the compact linear group \mathfrak{H} on the n -dimensional complex space B^+ is irreducible, or has only two invariant subspaces, the representations of \mathfrak{H} in which are inequivalent and mutually contragredient, then in B^+ there exists a subset B which is a real invariant and irreducible space with respect to \mathfrak{H} , of dimension n .

The representation of the algebra H^+ in the space L of all complex matrices by the formula

$$\psi(h)l = \varphi(h)l - l\varphi(h), \quad h \in H^+, l \in L,$$

is the Kronecker product of the representation φ by the representation $\tilde{\varphi}$, contragredient to φ . L is the direct sum of G^+ and the one-dimensional space of scalar matrices. Thus, if $\varphi(H) \subset G$ generates a pro-

space M with irreducible rotation group, then in the decomposition of the Kronecker product $\varphi(H^+) \times \tilde{\varphi}(H^+)$ into irreducibles, in addition to the adjoint representation and the zero representation acting in the space of scalar matrices, there will occur either a single irreducible representation, or only two irreducible non-equivalent mutually contragredient representations. Conversely, this is sufficient for the irreducibility of the rotation group in the space $\mathfrak{G}/\mathfrak{S}$.

3°. An irreducible representation φ of a semisimple algebra is completely characterized by its highest weight Λ . Any vector P of a Cartan subalgebra can be specified by a diagram, writing in the diagram of the algebra above each simple root α the number

$$P_\alpha = \frac{2(P, \alpha)}{(\alpha, \alpha)}$$

(the numerical mark). In order that a vector P be the highest weight Λ of an irreducible representation, it is necessary and sufficient that all its numerical marks be integral and nonnegative.

Let us list the highest weights of the adjoint representations of simple groups:

The numbers under the diagrams will be called the numbers of the simple roots. The diagrams Λ and $\tilde{\Lambda}$ of mutually contragredient representations for the alge-

bras A_m, D_m ($m \geq 4$), E_6 are related by the formulas:

$$\begin{aligned} \Lambda_{\alpha_i} &= \tilde{\Lambda}_{\alpha_{m-i+1}}, \quad i = 1, 2, \dots, m, \quad \text{for } A_m; \\ \Lambda_{\alpha_i} &= \tilde{\Lambda}_{\alpha_{6-i}}, \quad i = 1, 2, \dots, 5; \quad \Lambda_{\alpha_6} = \tilde{\Lambda}_{\alpha_6} \quad \text{for } E_6; \\ \Lambda_{\alpha_i} &= \tilde{\Lambda}_{\alpha_i}, \quad i = 1, 2, \dots, m-2; \quad \Lambda_{\alpha_{m-1}} = \tilde{\Lambda}_{\alpha_m}, \quad \Lambda_{\alpha_m} = \tilde{\Lambda}_{\alpha_{m-1}} \quad \text{for } D_m. \end{aligned}$$

All irreducible representations of the algebras $B_m, C_m, E_7, E_8, F_4, G_2$ are self-contragredient.

Following E. B. Dynkin ⁽¹⁾, we call a chain of simple roots of a semisimple algebra a collection $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$ of simple roots such that

$$(\alpha_{i_s}, \alpha_{i_{s+1}}) \neq 0, \quad s < k; \quad (\alpha_{i_p}, \alpha_{i_q}) = 0, \quad |p - q| > 1. \quad (5)$$

The chains of simple roots of the algebras A_m, B_m, C_m, F_4, G_2 have the form

$$\alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}, \quad r \geq 1, \quad s \geq 0, \quad r + s \leq m.$$

The chains of simple roots of the algebras D_m ($m \geq 4$), E_6, E_7, E_8 are as follows:

$$D_m : \text{I. } \alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}, \quad r \geq 1, \quad s \geq 0, \quad r + s \leq m - 1.$$

$$\text{II. } \alpha_r, \alpha_{r+1}, \dots, \alpha_{m-2}, \alpha_m, \quad 1 \leq r \leq m - 2. \quad \text{III. } \alpha_{m-1}, \alpha_{m-2}, \alpha_m.$$

$$E_m \quad (m = 6, 7, 8) : \text{I. } \alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}, \quad r \geq 1, \quad s \geq 0, \quad r + s \leq m - 1.$$

$$\text{II. } \alpha_r, \alpha_{r+1}, \dots, \alpha_3, \alpha_m, \quad 1 \leq r \leq 3. \quad \text{III. } \alpha_m, \alpha_3, \alpha_4, \dots, \alpha_l, \quad 3 \leq l \leq m - 1.$$

Theorem (E. B. Dynkin ⁽¹⁾). *Let φ and ψ be two irreducible representations of a complex semisimple algebra L of highest weights Λ and M , and let $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$ be a chain of simple roots such that*

$$(\Lambda, \alpha_{i_1}) \neq 0; \quad (\Lambda, \alpha_{i_s}) = 0, \quad s > 1; \quad (M, \alpha_{i_k}) \neq 0, \quad (M, \alpha_{i_s}) = 0, \quad s < k. \quad (6)$$

Then in the decomposition of the Kronecker product $\varphi \times \psi$ there will occur a representation of highest weight $\Lambda + M - S$, $S = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_k}$.

For two nonzero highest weights Λ and M of representations of simple algebras there always exists a chain with properties (6).

4°. Solution of problem A, if the embedding $\varphi(H^+) \subset G^+$ is irreducible.
In this case H^+ is necessarily semisimple.

1. First consider the case of a simple algebra H^+ . The decomposition of the Kronecker product $\varphi \times \tilde{\varphi}$ ($\tilde{\varphi}$ contragredient to φ) contains the components $\Lambda + \tilde{\Lambda}$ ($\tilde{\Lambda}$ is the highest weight of $\tilde{\varphi}$), the adjoint representation (highest weight T), $\Lambda + \tilde{\Lambda} - S$, where S is a chain of simple roots for Λ and $\tilde{\Lambda}$ with properties (6).

If $\varphi(H^+) \subset G^+$ is a solution of problem A, then one of the relations holds:

$$\Lambda + \tilde{\Lambda} - S = T, \tag{7}$$

$$\Lambda + \tilde{\Lambda} = T. \tag{8}$$

If (7) holds, then $T + S = \Lambda + \tilde{\Lambda}$ is self-contragredient and has nonnegative numerical labels. Among all the schemes of chains written out by us for A_m, B_m, C_m, F_4, G_2 , only the following satisfy this condition:

$$A_m : 1) r = 1, r + s = m; \quad 2) r = 2, r + s = m - 1, m \geq 3.$$

$$B_m : 1) r = 1, r + s = m; \quad 2) r = 1, s = 0, m \geq 3.$$

$$C_2 : 1) r = 2, s = 0.$$

This gives for $T + S = \Lambda + \tilde{\Lambda}$ the following possible schemes:

$$\text{(diagram for } A_m) \tag{9}$$

$$\text{(diagram for } A_m) \tag{10}$$

$$\text{(diagram for } A_m) \tag{11}$$

$$\text{(diagram for } B_m) \tag{12}$$

$$\text{(diagram for } B_m) \tag{13}$$

$$\text{(diagram for } C_2) \tag{14}$$

Hence Λ , up to contragredience, can only be the following:

$$\text{(diagram for } A_m, m \geq 1) \tag{15}$$

$$\text{(diagram for } A_m, m \geq 3) \tag{16}$$

$$\text{(diagram for } B_m, m \geq 2) \tag{17}$$

$$\text{(diagram for } A_m, m \geq 2) \tag{18}$$

The chains of simple roots of the algebras D_m ($m \geq 4$), E_6, E_7, E_8, F cannot satisfy relation (7), with the exceptions:

$$D_m (m > 4) : \alpha_1; \quad D_4 : 1) \alpha_1, 2) \alpha_3, 3) \alpha_4; \quad D_5 : 1) \alpha_4, \alpha_3, \alpha_5.$$

$$E_6 : \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5.$$

This leads to the following Λ (up to outer automorphisms):

$$\text{(three diagrams)} \tag{19}$$

Equality (8) gives (up to contragredience):

$$\Lambda \text{ (diagram)} \tag{20}$$

or

$$\Lambda \text{ (diagram)} \tag{21}$$

By counting dimensions we find that the solutions of problem A are the representations φ specified by the diagrams (15), (16), (17), (19), (21).

2. Let $H^+ = \sum_{i=1}^k H_i$ be not simple, but semisimple. An irreducible representation $\varphi(H^+)$ is specified by a diagram $(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$, where Λ_i ($i = 1, 2, \dots, k$) is the diagram of an irreducible representation of H_i .

In the decomposition $\varphi \times \tilde{\varphi}$ there enter the irreducible components of the adjoint representation of H^+ (their diagrams are: $(0, 0, \dots, 0, T_i, 0, \dots, 0)$, 0 is the diagram of the zero representation, T_i is the diagram of the adjoint representation of H_i) and the components

$$(\Lambda_1 + \tilde{\Lambda}_1, \Lambda_2 + \tilde{\Lambda}_2, \dots, \Lambda_{i-1} + \tilde{\Lambda}_{i-1}, \Lambda_i + \tilde{\Lambda}_i - S_i, \Lambda_{i+1} + \tilde{\Lambda}_{i+1}, \dots, \dots, \Lambda_k + \tilde{\Lambda}_k), \quad i = 1, 2, \dots, k.$$

Here S_i is a chain for Λ_i and $\tilde{\Lambda}_i$ with properties (6). Therefore, if φ is a solution of problem A, $k = 2$ and

$$\Lambda_i + \tilde{\Lambda}_i - S_i = 0,$$

i.e. φ , up to contragredience, is specified by the diagrams

$$\left(\begin{array}{ccccccc} 1 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ 1 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \end{array} \right) \quad \text{or} \quad \left(\begin{array}{ccccccc} 1 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ \circ & - & \circ & - & \dots & - & \circ & - & 1 & \circ \end{array} \right), \quad m_1 > 1, m_2 > 1. \tag{22}$$

m_i denotes the rank of H_i , $i = 1, 2$. The diagrams (22), as dimension counting shows, give a solution of problem A. If φ is reducible, the solutions of problem A possess a symmetric structure.

After eliminating the solutions of problem A of symmetric structure, we obtain the final result. All nonsymmetric homogeneous Riemannian spaces with irreducible rotation group and with motion group isomorphic to $SU(N)$, up to local isomorphism, are described by the following table of highest weights (for the embedding $\varphi(\mathfrak{G}) \subset SU(N)$):

	φ (m -rank \mathfrak{G})	Condition	N-dimension of φ	Dimension of the homogeneous space
1)	$\begin{array}{ccccccc} 2 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ \circ & - & \circ & & & & & \circ & - & \circ \end{array}$	$m \geq 2$	C_{m+2}^2	$\frac{m(m+1)^2(m+4)}{4}$
2)	$\begin{array}{ccccccc} \circ & - & 1 & \circ & - & \dots & - & \circ & - & \circ \\ \circ & & & \circ & - & \circ & & & & \circ \end{array}$	$m \geq 4$	C_{m+1}^2	$\frac{(m+1)^2(m+2)(m-2)}{4}$
3)	$\left(\begin{array}{ccccccc} 1 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ 1 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \end{array} \right)_{\frac{1}{1}}$	$\max(m_1, m_2) > 1$	$(m_1 + 1)(m_2 + 1)$	$(m_1^2 + 2m_1)(m_2^2 + 2m_2)$
4)	$\left(\begin{array}{ccccccc} 1 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ \circ & - & \circ & - & \dots & - & \circ & - & 1 & \circ \end{array} \right)_{\frac{1}{1}}$	$\min(m_1, m_2) > 1$	$(m_1 + 1)(m_2 + 1)$	$(m_1^2 + 2m_1)(m_2^2 + 2m_2)$
5)	$\begin{array}{ccccccc} \circ & - & \circ & - & \circ & & & \nearrow & 1 & \circ \\ & & & & & & & \searrow & \circ & \end{array}$		16	210
6)	$\begin{array}{ccccccc} 1 & \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ & & & & & & & & & \circ \end{array}$ with one additional node attached below the middle node		27	650

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1. E. B. Dynkin, *Tr. Moskovsk. matem. obshch.*, **1**, 39 (1952).

Note: Figure translations are in progress. See original paper for figures.

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