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Abstract

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EXISTENCE OF SOLUTIONS FOR A CERTAIN CLASS OF NONLINEAR INTEGRAL EQUATIONS

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Consider the equation

$$\varphi(x) = \int_0^1 K[x, y, \varphi(y)]\, dy, \tag{1}$$

where the function $K(x, y, z)$ is defined either on the set $P(A, B)$, or on P_{AB}, where $P(A, B)$ is defined by the inequalities: $0 \le x \le 1,\ 0 \le y \le 1,\ A < z < B$, and P_{AB} by the inequalities: $0 \le x \le 1,\ 0 \le y \le 1,\ -\infty < A \le z \le B < +\infty$.

We shall call equation (1) M-solvable if it has a solution and there exist constants C, D $(A < C < D < B)$ such that if $\varphi(x)$ is any solution of it, then $C < \varphi(x) < D$ for $0 \le x \le 1$.

In the present note several sufficient conditions for the M-solvability of equation (1) are given. The proofs of these conditions are based on the ideas of work (^{1}).

Lemma 1. *Let the function* $K(x, y, z)$ *be continuous on* P_{AB}*, the functions* $K_n(x, y, z)$ *converge to it uniformly on* P_{AB}*, and the equation*

$$\varphi(x) = \int_0^1 K_n[x, y, \varphi(y)]\, dy$$

has a solution for every n*. Then equation (1) also has a solution.*

Let I be the unit square $0 \le x \le 1,\ 0 \le y \le 1$ of the x, y-plane. Let $Q(A, B)$ and Q_{AB} be the sets of points of the y, z-plane defined respectively by the inequalities $0 \le y \le 1,\ A < z < B$, and $0 \le y \le 1,\ -\infty < A \le z \le B < +\infty$. Let Δ_i^m be the set of points of the x-axis defined by the inequality $\frac{i-1}{m} \le x < \frac{i}{m}$ for $i < m$, and by the inequality $\frac{m-1}{m} \le x \le 1$ for $i = m$.

We shall call a function $K(x, y, z)$ stepwise on a set P of the space (x, y, z) if it is continuous with respect to the variables y, z on P and if $K(x, y, z) = K\left(\frac{i-1}{m}, y, z\right)$ for $x \in \Delta_i,\ i = 1, 2, \dots, m,\ \{x, y, z\} \in P$.

Lemma 2. *Let the function* $K(x, y, z)$ *be stepwise on* P_{AB}, *the function* $f(y, z)$ *continuous on* Q_{AB},

$$\left[A - \int_0^1 f(y, A)\, dy\right] \cdot \left[B - \int_0^1 f(y, B)\, dy\right] < 0,$$

$$K^t(x, y, z) \equiv (1 - t)K(x, y, z) + tf(y, z)$$

for $\{x, y, z\} \in P_{AB}$, $0 \leq t \leq 1$. *Then, if every solution* $\varphi(x)$ *of the equation*

$$\varphi(x) = \int_0^1 K^t[x, y, \varphi(y)]\, dy \tag{2}$$

satisfies the inequality $A < \varphi(x) < B$, then equation (2) has a solution for any $0 \leq t \leq 1$.

Let the function $K(x, y, z)$ be defined on $P(-\infty, +\infty)$. Put, for $z > 0$,

$$K(z) = \sup |K(x, y, t)| \quad \text{for } \{x, y, t\} \in P_{-z,z}.$$

Lemma 3. Let, for $z > 0$, the function $g(z) \geq 0$ and

$$\overline{\lim}_{z \to +\infty} \frac{g(z)}{z} < 1.$$

Then there exist constants C, D $(-\infty < C < D < +\infty)$ such that, for any function $K(x, y, z)$ defined on $P(-\infty, +\infty)$ and such that, for $z > 0$, $K(z) \leq g(z)$, every solution $\varphi(x)$ of equation (1) satisfies the inequality $C < \varphi(x) < D$.

Proof. Suppose that the lemma is false. Then there exist functions

$$\varphi_n(x) = \int_0^1 K_n[x, y, \varphi_n(y)]\, dy,$$

where the functions $K_n(x, y, z)$ satisfy the conditions of the lemma, such that, putting $M_n = \sup_x |\varphi_n(x)|$, we obtain

$$\lim_{n \to +\infty} M_n = +\infty.$$

Since $g(M_n) \geq K_n(M_n) \geq |K_n[x, y, \varphi_n(y)]|$ for $\{x, y\} \in I$, $n = 1, 2, ...$, then for a sequence $\{\xi_n\}$ such that $|\varphi(\xi_n)| > M_n - M_n/n$, from (1) we can write

$$1 = \frac{\int_0^1 K_n[\xi_n, y, \varphi_n(y)]\, dy}{\varphi(\xi_n)} \le \frac{\int_0^1 |K_n[\xi_n, y, \varphi_n(y)]|\, dy}{|\varphi_n(\xi_n)|} \le \frac{g(M_n)}{M_n(1-1/n)},$$

whence, by virtue of the condition

$$\overline{\lim}_{z\to+\infty} \frac{g(z)}{z} < 1,$$

we obtain a contradiction.

Lemma 4. Let the function $K(x, y, z)$ be a step function on $P(-\infty, +\infty)$, and

$$\overline{\lim}_{z\to+\infty} \frac{K(z)}{z} < 1.$$

Then equation (1) is M-solvable.

Proof. Let

$$K^t(x, y, z) \equiv (1-t)K(x, y, z) + tK(0, y, z)$$

for $\{x, y, z\} \in P(-\infty, +\infty)$, $0 \le t \le 1$. Then $K^t(z) \le K(z)$ for $z > 0$, $0 \le t \le 1$. By Lemma 3 there exist constants C, D $(-\infty < C < D < +\infty)$ such that every solution $\varphi(x)$ of equation (2) satisfies the inequality $C < \varphi(x) < D$ for $0 \le x \le 1$, $0 \le t \le 1$. Using the condition

$$\overline{\lim}_{z\to+\infty} \frac{K(z)}{z} < 1,$$

we obtain

$$\left[C - \int_0^1 K(0, y, C)\, dy\right] < 0, \qquad \left[D - \int_0^1 K(0, y, D)\, dy\right] > 0.$$

Applying Lemma 2 to (2), we obtain the required result, since for $t = 0$ (2) becomes (1).

Theorem 1. Let the function $K(x, y, z)$ be continuous on $P(-\infty, +\infty)$ and

$$\overline{\lim}_{z\to+\infty} \frac{K(z)}{z} < 1.$$

Then equation (1) is M-solvable.

Proof. Let

$$K^m(x,y,z) \equiv K\left(\frac{i-1}{m}, y, z\right)$$

for $x \in \Delta_i$, $i = 1, 2, \dots, m$, $\{x, y, z\} \in P(-\infty, +\infty)$. The functions $K^m(x,y,z)$ converge to $K(x,y,z)$ uniformly on every P_{CD}, and $K^m(z) \leq K(z)$ for $z > 0$, $m = 1, 2, \dots$.

Using successively Lemmas 4, 3, 1, we obtain the M-solvability of equation (1).

Let the function $f(y,z) \geq 0$ be continuous on $Q(0, +\infty)$. Put

$$\overline{f}(z) = \sup \frac{f(y,z)}{z} \quad \text{for } 0 \leq y \leq 1, \qquad \underline{f}(z) = \inf \frac{f(y,z)}{z} \quad \text{for } 0 \leq y \leq 1.$$

We shall say,

that the function $f(y,z)$ is α_1, α_2-bounded, if either

$$\overline{\lim}_{z\to 0} f(z) < \frac{1}{\alpha_2}$$

and

$$\lim_{z\to +\infty} f(z) > \frac{1}{\alpha_1},$$

or

$$\lim_{z\to 0} f(z) > \frac{1}{\alpha_1} \quad \text{and} \quad \overline{\lim}_{z\to +\infty} f(z) < \frac{1}{\alpha_2},$$

where $\alpha_2 > \alpha_1 > 0$.

Lemma 5. Let the function $f(y,z)$ be α_1, α_2-bounded. Then there exist constants C, D $(0 < C < D < +\infty)$ such that, for every function $K(x,y,z)$ defined on $P(0, +\infty)$ and such that

$$\alpha_1 f(y,z) \leq K(x,y,z) \leq \alpha_2 f(y,z)$$

for $\{x,y,z\} \in P(0,+\infty)$, every solution $\varphi(x)$ of equation (1) satisfies the inequality

$$C < \varphi(x) < D.$$

Proof. Let

$$\varphi(x) = \int_0^1 K[x, y, \varphi(y)]\, dy,$$

where the function $K(x, y, z)$ satisfies the conditions of the lemma. Then

$$\alpha_1 \int_0^1 f[y, \varphi(y)]\, dy \leq \int_0^1 K[x, y, \varphi(y)]\, dy \leq \alpha_2 \int_0^1 f[y, \varphi(y)]\, dy. \tag{3}$$

Suppose that the lemma is false. Then there exist functions

$$\varphi_n(x) = \int_0^1 K_n[x, y, \varphi_n(y)]\, dy,$$

where the functions $K_n(x, y, z)$ satisfy the conditions of the lemma, such that, if

$$M_n = \sup_x \varphi_n(x), \qquad m_n = \inf_x \varphi_n(x),$$

then either

$$\lim_{n\to+\infty} M_n = +\infty,$$

or

$$\lim_{n\to+\infty} m_n = 0.$$

We restrict ourselves to the case

$$\lim_{n\to+\infty} M_n = +\infty.$$

From (3) we have

$$\lim_{n\to+\infty} m_n = +\infty.$$

Let

$$\lim_{z\to+\infty} f(z) > \frac{1}{\alpha_1}.$$

Then, for a sequence $\{\xi_n\}$ such that

$$\varphi(\xi_n) < m_n + \frac{m_n}{n},$$

using (1), (3), and the continuity of $f(y, z)$, we obtain

$$1 = \frac{\int_0^1 K_n[\xi_n, y, \varphi_n(y)]\, dy}{\varphi(\xi_n)} \geq \alpha_1 \frac{\int_0^1 f[y, \varphi_n(y)]\, dy}{m_n + m_n/n} \geq \alpha_1 \frac{f(z_n) z_n}{(1 + 1/n) m_n},$$

where $0 \leq y_n \leq 1$, $m_n \leq z_n \leq M_n$; whence, taking into account

$$\lim_{n\to+\infty} z_n = +\infty$$

and

$$\lim_{z\to+\infty} f(z) > \frac{1}{\alpha_1},$$

we obtain a contradiction. In the case

$$\overline{\lim}_{z\to+\infty} f(z) < \frac{1}{\alpha_2},$$

we take as $\{\xi_n\}$ a sequence such that

$$\varphi(\xi_n) > M_n - \frac{M_n}{n};$$

the subsequent arguments are analogous.

Lemma 6. Let the function $K(x, y, z)$ be stepwise on $P(0, +\infty)$, and

$$\alpha_1 f(y, z) \leq K(x, y, z) \leq \alpha_2 f(y, z)$$

for $\{x, y, z\} \in P(0, +\infty)$, where the function $f(y, z)$ is α_1, α_2-bounded. Then equation (1) is M-solvable.

Theorem 2. Let the function $K(x, y, z)$ be continuous on $P(0, +\infty)$, and

$$\alpha_1 f(y, z) \leq K(x, y, z) \leq \alpha_2 f(y, z)$$

for $\{x, y, z\} \in P(0, +\infty)$, where the function $f(y, z)$ is α_1, α_2-bounded. Then equation (1) is M-solvable.

Lemma 6 and Theorem 2 are proved analogously to Lemma 4 and Theorem 1, except that Lemma 5 is used instead of Lemma 3.

If, in equation (1),

$$K(x, y, z) = K(x, y) f(y, z),$$

then we obtain the equation

$$\varphi(x) = \int_0^1 K(x, y) f[y, \varphi(y)]\, dy, \tag{4}$$

and from Theorem 2 there follows Theorem 3.

Theorem 3. Let the function $K(x, y)$ be continuous on I, the function $f(y, z)$ continuous on $Q(0, +\infty)$, and

$$\overline{\lim}_{z\to+\infty} K \frac{g(z)}{z} < 1,$$

where

$$K = \sup_{\{x,y\}\in I} |K(x, y)|, \qquad g(z) = \sup_{\{y,t\}\in Q_{-z,z}} |f(y, t)|.$$

Then equation (4) is M-solvable.

Further, from Theorem 3, for $f(y, z) = z^\alpha$, it follows:

Theorem 4. *Let the function $K(x,y) > 0$ and be continuous on I. Then the equation*

$$\varphi(x) = \int_0^1 K(x,y)\varphi^\alpha(y)\,dy \tag{5}$$

for $\alpha \neq 1$ is M-solvable.

Let us note that for $\alpha = 1$, (5) is a Fredholm equation of the second kind and may either have or fail to have solutions under the conditions of Theorem 4. For $-1 \leq \alpha < 1$, the equation has a unique solution $\varphi(x)$, with $\varphi(x) = \lim_{n\to+\infty} \varphi_n(x)$, where $\varphi_0(x)$ is any positive function continuous for $0 \leq x \leq 1$, and

$$\varphi_{n+1}(x) = \varphi_n^{\frac{\alpha}{\alpha-1}}(x)\left[\int_0^1 K(x,y)\varphi_n^\alpha(y)\,dy\right]^{\frac{1}{1-\alpha}}.$$

This fact was proved by A. S. Kronrod for $-1 \leq \alpha \leq 0$. For $|\alpha| > 1$, for each α one can find an infinitely differentiable function $K(x,y)$ such that equation (5) has at least 3 solutions.

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References

1. J. Leray, J. Schauder, UMN, 1, no. 3-4 (13-14), 23 (1943).

Note: Figure translations are in progress. See original paper for figures.