ON INCONSISTENT SYSTEMS OF LINEAR INEQUALITIES

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Abstract

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ON INCONSISTENT SYSTEMS OF LINEAR INEQUALITIES

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This note considers inconsistent systems of linear inequalities from the point of view of approximating them by consistent systems; the concept of the defect of an arbitrary system is introduced, and a number of theorems are proved, from which, in particular, some known results follow as consequences.

1. Let

$$L_j(x) - a_j = \sum_{i=1}^{n} a_{ji}x_i - a_j \leqslant 0 \quad (j = 1, \dots, m)^* \tag{1}$$

be an arbitrary system of linear inequalities of rank $r > 0$, in which all coefficients a_{ji} and constant terms a_j are real numbers. We introduce for consideration the system

$$L_j(x) - a_j \leqslant \varepsilon_j \quad (j = 1, \dots, m), \tag{2}$$

in which ε_j are certain numerical parameters taking only such values for which system (2) is consistent. Defining the norm $\|\bar{\varepsilon}\|$ of the vector $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$ as $\max_j |\varepsilon_j|$, it is easy to see that the lower bound

$$\varepsilon_0 = \inf_{\bar{\varepsilon}} \|\bar{\varepsilon}\|$$

exists and is attained. This means, in particular, that the system of linear inequalities

$$L_j(x) - a_j \leqslant \varepsilon_0 \quad (j = 1, \dots, m) \tag{3}$$

is consistent. The number ε_0 will be called the **defect of system** (1).

The defect of system (1) can also be characterized as the lower bound of the nonnegative numbers ε for which the system $L_j(x) - a_j \leqslant \varepsilon$ $(j = 1, \dots, m)$ is consistent.

Of practical importance is the question of finding the defect of an arbitrary system of linear inequalities. We note that this question is connected with the minimax problem for a system of real linear functions $L_j(x) - a_j$, $1 \leqslant j \leqslant m$ (as well as with the problem of the least deviation of an arbitrary system of linear equations). Namely, the minimax of the system of functions under consideration is, as is easy to see, equal to the defect of the system of linear inequalities $L_j(x) - a_j \leqslant 0$, $-L_j(x) + a_j \leqslant 0$ $(j = 1, \dots, m)$.

In what follows we shall need certain definitions used in the theory of linear inequalities.

An inequality $L_k(x) - a_k \leqslant 0$ $(1 \leqslant k \leqslant m)$ of system (1), under the assumption that it is consistent, is called **unstable** if replacing it by the inequality $L_k(x) - a_k < 0$ destroys its consistency; otherwise the inequality $L_k(x) - a_k \leqslant 0$ is called stable (see ([1])).

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Everywhere below, by $L_j(x)$ we shall understand the linear form $\sum_{i=1}^{n} a_{ji} x_i$, with $L_j(x) \not\equiv 0$.

A **characteristic determinant** of system (1) is a determinant obtained by bordering an arbitrary nonzero minor of order r of its matrix by means of the column of constant terms of the system and a row of this matrix not belonging to the bordered minor.

Lemma 1. If, in a consistent system of linear inequalities $L_j(x) - a_j \leqslant 0$ $(j = 1, \dots, r+1)$ of rank $r > 0$, at least one inequality is unstable, then all its characteristic determinants are equal to zero.

The lemma is easily obtained by relying on Theorem 22 of [1].

Let

$$L_j(x) - a_j \leqslant 0 \qquad (j = 1, 2, \dots, r+1) \tag{4}$$

be a system of linear inequalities of rank $r > 0$; let Δ be an arbitrary characteristic determinant of it, and let A_i be the algebraic cofactor of the last element of the i-th row of the determinant Δ.

Theorem 1. If system (4) is inconsistent, then its defect ε_0 is equal to

$$-\left(\Delta : \sum_{1}^{r+1} A_i\right).$$

Indeed, since at least one inequality of the system $L_j(x) - a_j \leqslant \varepsilon_0$ $(j = 1, \dots, r+1)$ is unstable, the assertion is easily obtained by applying Lemma 1.

Noting that a system of linear inequalities is inconsistent if and only if its defect is greater than zero, from Theorem 1 we obtain:

Corollary 1. System (4) is inconsistent if and only if

$$\Delta \Big/ \sum_{1}^{r+1} A_i < 0.$$

The inequality

$$\Delta \Big/ \sum_{1}^{r+1} A_i \geqslant 0$$

gives a necessary and sufficient condition for its consistency.

Remark. The sufficiency of the condition

$$\Delta \Big/ \sum_{1}^{r+1} A_i \geqslant 0$$

for the consistency of system (4) can be discerned from Corollary 3 of Theorem 3 of [1].

Lemma 2. Let system (4) be inconsistent and let ε_0 be its defect, and let Δ be an arbitrary characteristic determinant of it. Then the k-th inequality of the system $L_j(x) - a_j \leqslant 0$ $(j = 1, \dots, r+1)$ is stable if $A_k = 0$, and unstable if $A_k \neq 0$.

The lemma follows from Theorem 22 of [1], taking into account the fact that the nonzero algebraic cofactors A_k, determined by the characteristic determinant Δ, have the same signs.

Theorem 2. The set of solutions of the system of linear inequalities $L_j(x) - a_j \leqslant \varepsilon_0$ $(j = 1, \dots, r+1)$* coincides with the set of solutions of the system of linear equations $L_j(x) - a_j = \alpha_j$ $(j = 1, \dots, r+1)$, where α_j are numerical parameters, with $\alpha_j = \varepsilon_0$ for $A_j \neq 0$ and $\alpha_j \leqslant \varepsilon_0$ for $A_j = 0$.

The theorem is easily obtained by taking into account Lemma 2 and the fact that the system of linear equations written above, under the assumptions made with respect to α_j, is always consistent.

2. **Theorem 3.** The defect of an arbitrary inconsistent system of linear inequalities $L_j(x) - a_j \leqslant 0$ $(j = 1, \dots, m)$ of rank $r > 0$ is equal to the greatest of the defects of its inconsistent subsystems of rank r consisting of $r+1$ inequalities, determined in accordance with Theorem 1.

The proof is carried out using Corollary 4 of Theorem 1 from [2].

Remark. Theorems 1 and 3 indicate an algorithm (though, in the general case, a cumbersome one) also for actually finding the defect of an arbitrary inconsistent system of linear inequalities. The defect ε_0 can

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* It is assumed that, for the system $L_j(x) - a_j \leqslant \varepsilon_0$ $(j = 1, \dots, r+1)$, the conditions of Lemma 2 are satisfied.

can also be computed by the usual methods of linear programming, since it coincides with the least value of the function $f(x) \equiv x_{n+1}$ on the set of solutions of the system $L_j(x) - x_{n+1} - a_j \leqslant 0$ $(j = 1, \dots, m)$.

Corollary 2*. The minimax of a system of nonconstant linear functions $L_j(x) - a_j$ $(j = 1, \dots, m)$ of rank $r > 0$ is equal to the minimax of some subsystem of it of rank r, consisting of $r+1$ functions (or, in another formulation: the least deviation of an inconsistent system of linear equations of rank r, $L_j(x) - a_j = 0$ $(j = 1, \dots, m)$, $L_j(x) \not\equiv 0$, coincides with the least deviation of some subsystem of it of rank r, consisting of $r+1$ equations).

Indeed, the minimax of the system of functions $L_j(x) - a_j$ $(j = 1, \dots, m)$ is realized on some subsystem, determined by Theorem 3, of the system $L_j(x) - a_j \leqslant 0$, $-L_j(x) + a_j \leqslant 0$ $(j = 1, \dots, m)$. Let, for simplicity, this subsystem be $L_j(x) - a_j \leqslant 0$ $(j = 1, \dots, r+1)$. If to it one adjoins the inequalities $-(L_j(x) - a_j) \leqslant 0$ $(j = 1, \dots, r+1)$, then the defect of the subsystem thus obtained is again equal to ε_0; on the other hand, it coincides with the minimax of the linear functions $L_j(x) - a_j$ $(j = 1, \dots, r+1)$, as was required.

Let $L_j(x) - a_j$ $(j = 1, \dots, r+1)$ be a system of linear functions of rank r; let Δ be the characteristic determinant corresponding to the system of inequalities $L_j(x) - a_j \leqslant 0$ $(j = 1, \dots, r+1)$; and let A_i be the cofactor of the last element of the i-th row of the determinant Δ.

Corollary 3. The minimax E of the system of linear functions $L_j(x) - a_j$ $(j = 1, \dots, r+1)$ is equal to

$$|\Delta| \Big/ \sum_{1}^{r+1} |A_i|.$$

Indeed, the minimax E is equal to the defect of the system $\delta_j(L_j(x) - a_j) \leqslant 0$ $(j = 1, \dots, r+1)$, where δ_j is equal to 1 or -1. But the latter coincides with

$$-\Delta' \Big/ \sum_{1}^{r+1} A_i',$$

where Δ' is the determinant obtained from Δ by replacing the elements of an arbitrary j-th row by their opposites if $\delta_j = -1$, and A'_i is the cofactor of the last element of the i-th row of the determinant Δ'. Obviously, $|\Delta| = |\Delta'|$, $|A_i| = |A'_i|$ $(i = 1, \dots, r+1)$. Since the system (6) is inconsistent, all nonzero A'_i have the same signs. Therefore

$$E = \left| -\Delta' \Big/ \sum_1^{r+1} A'_i \right| = |\Delta'| \Big/ \sum_1^{r+1} |A'_i| = |\Delta| \Big/ \sum_1^{r+1} |A_i|.$$

The points of the space R_n whose coordinates satisfy the system (3) will be called the **points of best approximation of system** (1). For the case of a system of rank $r > 0$, consisting of $r+1$ inequalities, the question of finding the points of best approximation is solved by Theorem 2. Below it is solved for systems with a certain additional condition, which for systems of linear equations bears the name of the Haar condition (see ([5])).

We shall say that a system of linear inequalities of rank r **satisfies the Haar condition** if in its matrix of coefficients at the unknowns there exist r such columns that, in the matrix formed from them, all minors of order r are nonzero.

Theorem 4. Let

$$L_j(x) - a_j \leqslant 0 \qquad (j = 1, \dots, m) \tag{*}$$

be an inconsistent system of rank r, satisfying the Haar condition, and let $L_{j_s}(x) - a_{j_s} \leqslant 0$ $(s = 1, \dots, r+1)$ be its subsystem on which the de-

* This result is contained in ([3]).
** This formula is contained in ([4]).

the effect ε_0 of the system $(*)$. Then the system of linear inequalities $L_j(x) - a_j \leqslant \varepsilon_0$ $(j = 1, \dots, m)$ is equivalent to the system of linear equations $L_{j_s}(x) - a_{j_s} = \varepsilon_0$ $(s = 1, \dots, r)$.

The proof is carried out using Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.