

Soviet-era science, translated into English

Corresponding Member of the Academy of Sciences of the USSR V. V. SOBOLEV

1961

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.48866>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICAL PHYSICS

Corresponding Member of the Academy of Sciences of the USSR V. V. SOBOLEV

DIFFUSION OF RADIATION IN A MEDIUM WITH A SPECULARLY REFLECTING BOUNDARY

In the author's note ⁽¹⁾, the problem of the diffusion of radiation in a semi-infinite medium consisting of plane-parallel layers and bounded by a specular surface with an internal reflection coefficient equal to unity was considered. This problem reduces to the equation

$$B(\tau) = \frac{\lambda}{2} \int_0^\infty [\text{Ei}|\tau - t| + \text{Ei}(\tau + t)]B(t) dt + g(\tau). \quad (1)$$

Here $B(\tau)$ is the ratio of the radiation coefficient to the absorption coefficient (at optical depth τ); $g(\tau)$ is the ratio of the radiation coefficient due directly to sources of radiation to the absorption coefficient; λ is the probability of survival of a quantum in an elementary scattering event.

Equation (1) can be solved by using the Fourier transform. However, in the note mentioned, another method, proposed earlier by the author ⁽²⁾, was applied to solve this equation.

In the present note the same method is applied to the solution of an equation that is a generalization of equation (1). Namely, we assume that the reflection coefficient depends on the angle of incidence. In this case, instead of equation (1) we have

$$B(\tau) = \frac{\lambda}{2} \int_0^\infty [\text{Ei}|\tau - t| + K(\tau + t)]B(t) dt + g(\tau), \quad (2)$$

where

$$K(\tau) = \int_0^1 r(\zeta)e^{-\tau/\zeta} \frac{d\zeta}{\zeta}; \quad (3)$$

$r(\zeta)$ is the reflection coefficient; ζ is the cosine of the angle of incidence ($0 \leq r(\zeta) \leq 1$).

Along with equation (12) we consider the equation

$$B^*(\tau) = \frac{\lambda}{2} \int_0^\infty [\text{Ei}|\tau - t| - K(\tau + t)]B^*(t) dt + g(\tau). \quad (4)$$

Denoting the resolvents of equations (2) and (4), respectively, by $\Gamma(\tau, t)$ and $\Gamma^*(\tau, t)$, and proceeding in the same way as before ^(1,2), we obtain the following equations for determining the resolvents:

$$\begin{aligned} \frac{\partial \Gamma}{\partial \tau} + \frac{\partial \Gamma^*}{\partial t} &= \Phi^*(\tau)\Phi(t), \\ \frac{\partial \Gamma^*}{\partial \tau} + \frac{\partial \Gamma}{\partial t} &= \Phi(\tau)\Phi^*(t), \end{aligned} \quad (5)$$

where $\Phi(\tau) = \Gamma(0, \tau)$, $\Phi^*(\tau) = \Gamma^*(0, \tau)$. Thus, the problem is reduced to finding the functions $\Phi(\tau)$ and $\Phi^*(\tau)$.

We introduce two auxiliary equations by putting $g(\tau) = e^{-\tau/\zeta}$ in (2) and (4). Denoting the solutions of these equations respectively by $B(\tau, \zeta)$ and $B^*(\tau, \zeta)$, we obviously have

$$\begin{aligned} \Phi(\tau) &= \frac{\lambda}{2} \int_0^1 [1 + r(\zeta)]B(\tau, \zeta) \frac{d\zeta}{\zeta}, \\ \Phi^*(\tau) &= \frac{\lambda}{2} \int_0^1 [1 - r(\zeta)]B^*(\tau, \zeta) \frac{d\zeta}{\zeta}. \end{aligned} \quad (6)$$

Further, from the auxiliary equations we obtain

$$\begin{aligned} \frac{\partial B(\tau, \zeta)}{\partial \tau} &= -\frac{1}{\zeta}B(\tau, \zeta) + B(0, \zeta)\Phi^*(\tau), \\ \frac{\partial B^*(\tau, \zeta)}{\partial \tau} &= -\frac{1}{\zeta}B^*(\tau, \zeta) + B^*(0, \zeta)\Phi(\tau). \end{aligned} \quad (7)$$

We introduce the function

$$\rho(\eta, \zeta) = \int_0^\infty B(\tau, \zeta)e^{-\tau/\eta} \frac{d\tau}{\eta\zeta}, \quad (8)$$

which characterizes the intensity of radiation diffusely reflected by the medium when it is illuminated by parallel rays.

Using equations (7), we find

$$\rho(\eta, \zeta) = \frac{B(0, \eta)B^*(0, \zeta)\eta - B^*(0, \eta)B(0, \zeta)\zeta}{\eta^2 - \zeta^2}. \quad (9)$$

To determine the functions $B(0, \eta)$ and $B^*(0, \zeta)$, we obtain the equations

$$B(0, \eta) = 1 + \frac{\lambda}{2}\eta \int_0^1 [1 + r(\zeta)] \frac{B(0, \eta)B^*(0, \zeta)\eta - B^*(0, \eta)B(0, \zeta)\zeta}{\eta^2 - \zeta^2} d\zeta,$$

$$B^*(0, \eta) = 1 + \frac{\lambda}{2}\eta \int_0^1 [1 - r(\zeta)] \frac{B^*(0, \eta)B(0, \zeta)\eta - B(0, \eta)B^*(0, \zeta)\zeta}{\eta^2 - \zeta^2} d\zeta. \quad (10)$$

The problem of the diffuse reflection of light by a medium bounded by a mirror surface was considered earlier by V. A. Ambartsumian and by the author (see ⁽³⁾, Ch. VII). Equations (9) and (10) coincide with the equations of V. A. Ambartsumian, but differ from the author's equations.

Applying the device indicated earlier (see ⁽³⁾, Ch. IV), one can also obtain new equations for determining the functions $B(0, \eta)$ and $B^*(0, \eta)$, namely

$$B(0, \eta)a(\eta) = 1 + \frac{\lambda}{2}\eta \int_0^1 \frac{B(0, \zeta)}{\eta + \zeta} r(\zeta) d\zeta - \frac{\lambda}{2}\eta \int_0^1 \frac{B(0, \zeta)}{\eta - \zeta} d\zeta, \quad (11)$$

where

$$a(\eta) = 1 - \frac{\lambda}{2}\eta \ln \frac{1 + \eta}{1 - \eta} \quad (12)$$

(and an analogous equation for $B^*(0, \eta)$, differing from (11) by the sign before $r(\zeta)$).

Considering the functions $B(0, \eta)$ and $B^*(0, \eta)$ to be known, we can find the functions $\Phi(\tau)$ and $\Phi^*(\tau)$, and hence also the resolvents of the integral equations (2) and (4).

We apply the Laplace transform to equations (7) and use relations (6). Denoting

$$\int_0^\infty \Phi(\tau)e^{-s\tau} d\tau = \bar{\Phi}(s), \quad \int_0^\infty \Phi^*(\tau)e^{-s\tau} d\tau = \bar{\Phi}^*(s), \quad (13)$$

we obtain

$$\bar{\Phi}(s) = [1 + \bar{\Phi}(s)] \frac{\lambda}{2} \int_0^1 \frac{B^*(0, \eta)}{1 - s^2\eta^2} [1 + r(\eta)] d\eta - [1 + \bar{\Phi}^*(s)] \frac{\lambda}{2} \int_0^1 \frac{s\eta B(0, \eta)}{1 - s^2\eta^2} [1 - r(\eta)] d\eta, \quad (14)$$

$$\bar{\Phi}^*(s) = [1 + \bar{\Phi}^*(s)] \frac{\lambda}{2} \int_0^1 \frac{B(0, \eta)}{1 - s^2 \eta^2} [1 - r(\eta)] d\eta - [1 + \bar{\Phi}^*(s)] \frac{\lambda}{2} \int_0^1 \frac{s\eta B^*(0, \eta)}{1 - s^2 \eta^2} [1 + r(\eta)] d\eta.$$

From (14), with the aid of (10), we find

$$[1 + \bar{\Phi}^*(s)] \left(1 - \lambda \int_0^1 \frac{d\eta}{1 - s^2 \eta^2} \right) = 1 - \frac{\lambda}{2} \int_0^1 \frac{B(0, \eta)}{1 - s\eta} d\eta + \frac{\lambda}{2} \int_0^1 \frac{B(0, \eta)}{1 + s\eta} r(\eta) d\eta. \quad (15)$$

To compute $\Phi(\tau)$ by the formula

$$\Phi(\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{\Phi}(z) e^{z\tau} dz \quad (16)$$

one must take into account that the function $\bar{\Phi}(z)$ has a branch point $z = -1$ and a pole $z = -k$, where k is determined from the equation

$$\frac{\lambda}{2k} \ln \frac{1+k}{1-k} = 1. \quad (17)$$

Also taking relation (11) into account, by contour integration we obtain

$$\Phi(\tau) = C(k) e^{-k\tau} + 2\lambda \int_1^\infty \frac{x e^{-x\tau} A(1/x) dx}{(\lambda\pi)^2 + (2x + \lambda \ln \frac{x-1}{x+1})^2}, \quad (18)$$

where

$$C(k) = \frac{k(1-k^2)}{\lambda + k^2 - 1} \left\{ 1 - \frac{\lambda}{2} \int_0^1 \frac{B(0, \xi)}{1 + k\xi} d\xi + \frac{\lambda}{2} \int_0^1 \frac{B(0, \xi)}{1 - k\xi} r(\xi) d\xi \right\}, \quad (19)$$

$$A(\eta) = 1 + r(\eta) - \frac{\lambda}{2} \eta \int_0^1 \frac{B(0, \xi)}{\eta + \xi} [1 - r(\eta)r(\xi)] d\xi - \frac{\lambda}{2} \eta \int_0^1 \frac{B(0, \xi)}{\eta - \xi} [r(\eta) - r(\xi)] d\xi. \quad (20)$$

Here it is assumed that the function $A(\eta)$ has no singularities preventing the transition from (16) to (18).

The expression for the function $\Phi^*(\tau)$ is obtained from (18) by replacing $r(\zeta)$ by $-r(\zeta)$. Let us note two special cases of formula (18). In the absence of internal reflection ($r = 0$), we obtain

$$C(k) = \frac{k(1-k^2)}{\lambda+k^2-1} \left[1 - \frac{\lambda}{2} \int_0^1 \frac{\varphi(\zeta)}{1+k\zeta} d\zeta \right] = \left[\frac{\lambda}{2} \int_0^1 \frac{\varphi(\zeta)\zeta}{(1-k\zeta)^2} d\zeta \right]^{-1}, \quad (21)$$

$$A(\eta) = 1 - \frac{\lambda}{2} \eta \int_0^1 \frac{\varphi(\zeta)}{\eta+\zeta} d\zeta = \frac{1}{\varphi(\eta)}, \quad (22)$$

where $\varphi(\eta)$ is the Ambartsumian function. Substituting (21) and (22) into (18), we arrive at the formula previously found by I. N. Minin ⁽⁴⁾.

In the case of total internal reflection ($r = 1$), we have

$$C(k) = \frac{2k(1-k^2)}{\lambda+k^2-1}, \quad A(\eta) = 2. \quad (23)$$

Substituting (23) into (18), we obtain the formula previously found by the author ⁽¹⁾.

Equations of type (2) may arise in various problems. We indicate two of them.

1. Let the diffusion of radiation in a spectral line occur in an expanding planetary nebula or in the envelope of a nova. For simplicity, we shall assume that the envelope has spherical symmetry and that its thickness is small in comparison with the distance from the center of the star. We shall also assume that the scattering of radiation is coherent and that the profile of the absorption coefficient is rectangular. If the envelope is stationary, then the intensity of the radiation emerging through the inner boundary of the envelope is equal to the intensity of the radiation arriving from the opposite side of the envelope (i.e., "total internal reflection" takes place). In this case the problem reduces to equation (1). If, however, the envelope is expanding, then, owing to the Doppler effect, only part of the quanta is "reflected" from the opposite side of the envelope. In this case the problem reduces to equation (2), where $r(\zeta) = 1 - \frac{v}{u}\zeta$ for $\frac{v}{u}\zeta < 1$ and $r(\zeta) = 0$ for $\frac{v}{u}\zeta > 1$ (v is the expansion velocity of the envelope, u is the mean thermal velocity of the atoms). It is of interest to generalize equation (2) to the case of incoherent scattering.
2. Let the diffusion of radiation occur in a water basin. This process is described by equation (2), in which $r(\zeta)$ is given by the Fresnel formula. To approximate real conditions, it is of interest to generalize equation (2) to the case of anisotropic scattering.

Received
28 X 1960

CITED LITERATURE

- ¹ V. V. Sobolev, *DAN*, **129**, No. 6 (1959).
- ² V. V. Sobolev, *DAN*, **116**, No. 1 (1957); **120**, No. 1 (1958); *Izv. AN ArmSSR, ser. fiz.-matem. nauk*, **11**, No. 5 (1958); *Astr. zhurn.*, **36**, No. 4 (1959).
- ³ V. V. Sobolev, *Transfer of Radiant Energy in the Atmospheres of Stars and Planets*, Moscow, 1956.
- ⁴ I. N. Minin, *DAN*, **120**, No. 1 (1958).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.