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Abstract

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MATHEMATICS

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SOME TRANSFORMATIONS OF THE FUNDAMENTAL EQUATIONS OF THE THEORY OF SURFACES

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§ 1°. On a surface of constant negative curvature the angle ω between the asymptotic lines satisfies the well-known equation

$$\frac{\partial^2\omega}{\partial v \partial u} = K \sin\omega, \tag{1}$$

where u, v are natural parameters of the asymptotic lines; K is the Gaussian curvature ($=$ const < 0).

In arbitrary asymptotic parameters u, v, equation (1) has the form

$$\frac{\partial^2\omega}{\partial s_2 \partial s_1} = K \sin\omega, \tag{2}$$

where

$$\frac{\partial}{\partial s_1} = \frac{1}{\sqrt{E}} \frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial s_2} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial v}.$$

We show in this note that also *in the case of a surface of variable negative curvature the angle ω, referred to asymptotic parameters, satisfies an equation in many respects similar to equation* (2) *(see § 5°, equation (III)).* This equation can be written symbolically in the form

$$L_{s_2} L^*_{s_1}(\omega) = M \sin\omega, \tag{3}$$

where L_{s_2} is a linear differential operator analogous to the operator $\partial/\partial s_2$; $L^*_{s_1}$ is a quasilinear operator, basically of the same type; $L_{s_2} L^*_{s_1}$ does not depend on the choice of asymptotic parameters (although the operators L_{s_2}, $L^*_{s_1}$ separately do not possess such an invariance property). The letter M in equation (3) denotes a quantity almost belonging to the intrinsic geometry of the surface; at least,

M admits estimates under certain conditions imposed only on the metric. An approach to an equation of the form (3) was given in the note of N. V. Efimov (^{1}). An application of this equation will be given in the following note by the authors, which will be published in this same journal.

§ 2°. Let us agree on notation and orientation. By K we denote the Gaussian curvature; along with this we use the notation

$$k^2 = |K|, \qquad Q = \ln \sqrt{k}.$$

We write the linear element in arbitrary asymptotic parameters u, v:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2;$$

the asymptotic direction $v = \text{const}$ is considered the first one. We also put $E = e^2$, $G = g^2$, $W^2 = EG - F^2$. If l, m, n are the coefficients of the second quadratic-

normal form divided by W, then $l = 0,\ n = 0,\ m = \pm k$. We shall choose $m = +k$; thereby the normal is oriented consistently with the positive directions of the asymptotic lines (i.e., with the directions of increase of u, v), and these directions are determined up to their simultaneous reversal. We regard as positive rotations from the first asymptotic direction to the second. By ω we denote the angle between the positive asymptotic directions; $0 < \omega < \pi$. We set

$$\frac{\partial}{\partial s_1} = \frac{1}{e}\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial s_2} = \frac{1}{g}\frac{\partial}{\partial v};$$

if $\dfrac{\partial}{\partial s}$ denotes differentiation in a given direction, then $\dfrac{\partial}{\partial s^*}$ denotes differentiation in the direction obtained by rotating the given one through the angle $+\dfrac{\pi}{2}$. The symbol $\left(\dfrac{\partial^2}{\partial s^2}\right)_g$ denotes the second differentiation along the arc of the geodesic drawn in the given direction.

§ 3°. Let us give some auxiliary formulas. From the equalities $l = n = 0,\ m = k$ and from the Peterson-Codazzi equations we have:

$$\Gamma^1_{12} = -Q'_v, \qquad \Gamma^2_{12} = -Q'_u. \tag{4}$$

Hence

$$\frac{\partial e}{\partial s_2} = -e\left(\frac{\partial Q}{\partial s_2} + \cos\omega\,\frac{\partial Q}{\partial s_1}\right), \tag{5a}$$

$$\frac{\partial g}{\partial s_1} = -g\left(\frac{\partial Q}{\partial s_1} + \cos\omega\,\frac{\partial Q}{\partial s_2}\right). \tag{5b}$$

From (5 a, b) and from the relations

$$\cos\omega\,\frac{\partial Q}{\partial s_1}=\frac{\partial Q}{\partial s_2}-\sin\omega\,\frac{\partial Q}{\partial s_1^*},\tag{6a}$$

$$\cos\omega\,\frac{\partial Q}{\partial s_2}=\frac{\partial Q}{\partial s_1}+\sin\omega\,\frac{\partial Q}{\partial s_2^*}\tag{6b}$$

we obtain

$$\frac{\partial\ln(ek)}{\partial s_2}=\sin\omega\,\frac{\partial Q}{\partial s_1^*},\tag{7a}$$

$$\frac{\partial\ln(gk)}{\partial s_1}=-\sin\omega\,\frac{\partial Q}{\partial s_2^*}.\tag{7b}$$

In addition, the following expressions for the geodesic curvatures of the asymptotic lines hold:

$$\frac{1}{\rho_1}=-\frac{\partial\omega}{\partial s_1}+\sin\omega\,\frac{\partial Q}{\partial s_2},\tag{8a}$$

$$\frac{1}{\rho_2}=+\frac{\partial\omega}{\partial s_2}-\sin\omega\,\frac{\partial Q}{\partial s_1}.\tag{8b}$$

If Φ is any scalar field on the surface, then its second derivative with respect to the arc of an arbitrary line γ drawn through the given point in the given direction is expressed by the formula

$$\frac{\partial^2\Phi}{\partial s^2}=\left(\frac{\partial^2\Phi}{\partial s^2}\right)_g+\frac{1}{\rho_g}\frac{\partial\Phi}{\partial s^*},\tag{9}$$

where $\frac{1}{\rho_g}$ is the geodesic curvature of the line γ.

p. 4°. The formula is known (see (^{2}), p. 170)

$$K=\frac{1}{W}\left\{\frac{\partial}{\partial v}\left(\frac{W}{E}\Gamma_{11}^2\right)-\frac{\partial}{\partial u}\left(\frac{W}{E}\Gamma_{12}^2\right)\right\}.\tag{10}$$

Using the relation

$$-\frac{\partial\omega}{\partial u}=\frac{W}{E}\Gamma_{11}^2+\frac{W}{G}\Gamma_{12}^1,$$

we obtain from (10) and (4)

$$\omega''_{uv}-\left(\frac{g}{e}\sin\omega\,Q'_u\right)'_u-\left(\frac{e}{g}\sin\omega\,Q'_v\right)'_v=k^2eg\sin\omega.\tag{11}$$

From (11), applying (5), (6), (8), (9), we find

$$k^{3/2}\frac{\partial}{\partial s_2}\left[k^{-3/2}\left(\frac{\partial \omega}{\partial s_1}-\sin\omega\,\frac{\partial Q}{\partial s_2}\right)\right]=\left[k^2+A-3\frac{\partial Q}{\partial s_1^*}\frac{\partial \omega}{\partial s_1}\right]\sin\omega, \tag{I}$$

where

$$A=\left(\frac{\partial^2 Q}{\partial s_1^2}\right)_g-\left(\frac{\partial Q}{\partial s_1}\right)^2+3\sin\omega\,\frac{\partial Q}{\partial s_2}\frac{\partial Q}{\partial s_1^*}.$$

Taking (8a) into account, equation (I) can be given a more geometric form:

$$k^{3/2}\frac{\partial}{\partial s_2}\left(k^{-3/2}\frac{1}{\rho_1}\right)+\left[k^2+\left(\frac{\partial^2 Q}{\partial s_1^2}\right)_g-\left(\frac{\partial Q}{\partial s_1}\right)^2+3\frac{1}{\rho_1}\frac{\partial Q}{\partial s_1^*}\right]\sin\omega=0. \tag{II}$$

p. 5°. From (I), with the aid of (7a), there follows the equation which we also had in mind at the beginning of the note:

$$\frac{k^{3/2}}{(ek)^3}\frac{\partial}{\partial s_2}\frac{(ek)^3\left[\frac{\partial \omega}{\partial s_1}-\sin\omega\,\frac{\partial Q}{\partial s_2}\right]}{k^{3/2}}=\left[k^2+B\right]\sin\omega, \tag{III}$$

where

$$B=\left(\frac{\partial^2 Q}{\partial s_1^2}\right)_g-\left(\frac{\partial Q}{\partial s_1}\right)^2.$$

p. 6°. Proceeding in a somewhat different way, equation (II) can be obtained from Liouville's well-known formula for Gaussian curvature, and then (I) or (III) can be derived from equation (II). It should be noted that, despite the greater geometric character of equation (II), it is less convenient for applications than (I) and (III) (since it involves two quantities of extrinsic-geometric character: $\frac{1}{\rho_1}$ and ω).

With the aid of equation (I), one can somewhat strengthen the assertion about the correctness of Hilbert' s well-known theorem, stated in note (^{1}); namely, if in note (^{1}) it is asserted in essence that a slow variation of the Gaussian curvature entails a rapid variation of ω, then from equation (I) there follows the necessity of a rapid variation of ω in some asymptotic direction. More substantial applications of equations (I), (III) are given in the authors' following note.

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[2] L. Bianchi, *Lezioni di Geom. diff.*, 1, p. 1, 1927.

Note: Figure translations are in progress. See original paper for figures.

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