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Abstract

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CONVEX HOMOGENEOUS DOMAINS

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It is known that the n-dimensional ball is homogeneous with respect to a certain group of projective transformations: it is one of the models of n-dimensional Lobachevsky space. One may pose the question: what other convex domains are homogeneous with respect to projective transformations? Here two such domains which are carried into one another by a projective transformation should be regarded as equivalent.

The problem posed is in many respects analogous to, and has direct connections with, the problem of classifying bounded domains in n-dimensional complex space that are homogeneous with respect to analytic automorphisms, a problem which I. I. Pyatetskii-Shapiro has recently been successfully studying (see, in particular, ([3])). I take this opportunity to express my deep gratitude to I. I. Pyatetskii-Shapiro, who had a great influence on my work.

Projectively homogeneous convex n-dimensional domains are in one-to-one correspondence with convex $(n+1)$-dimensional cones homogeneous with respect to linear transformations. Namely, to each such domain $D \subset R^n$ there corresponds the cone $V \subset R^{n+1} \simeq R^n \times R^1$, consisting of the vectors

$$(x, t), \qquad x \in R^n, \qquad t \in R^1,$$

for which

$$tx \in D, \qquad t > 0. \tag{1}$$

Equivalent domains correspond to cones carried into one another by linear transformations.

In studying convex homogeneous cones one may restrict oneself to cones that contain no straight line. They correspond to projectively homogeneous domains equivalent to bounded ones. In what follows, by a convex cone we shall always mean a convex cone containing no straight line.

Thus, what kinds of convex homogeneous cones are there? In my note ([1]) all self-adjoint convex homogeneous cones were found. In the present note an approach is given to the study of all convex homogeneous cones, among which the self-adjoint cones constitute, in a certain sense, an exception.

It turns out to be reasonable to broaden the problem somewhat. We shall consider not only convex cones, but also arbitrary convex domains (not containing a straight line). If D is such a domain, then by $\mathfrak{G}(D)$ we shall denote the group of all affine transformations preserving the domain D. We shall call the domain D homogeneous (not to be confused with projectively homogeneous!) if the group $\mathfrak{G}(D)$ is transitive on D. In the particular case where D is a cone, the transformations in $\mathfrak{G}(D)$ automatically turn out to be linear: they preserve the vertex of the cone. Homogeneous domains D_1 and D_2 will be called equivalent if they are carried into one another by affine transformations.

With each n-dimensional convex homogeneous domain D, by formulas (1), there is associated an $(n+1)$-dimensional convex homogeneous cone $V = V(D)$, which we shall call the cone **spanned** over the domain D. At the same time it is not excluded, in contrast to the projective case, that equivalent cones correspond to nonequivalent domains D. A priori it is also unclear whether every convex homogeneous cone V is spanned over some convex homogeneous domain. Below it will be shown that this is so. It can also be shown that there exists only a finite number of nonequivalent homogeneous domains D for which $V \simeq V(D)$.

In what follows, D will always denote a fixed convex homogeneous domain in R^n.

Theorem 1. *The stationary subgroup $\mathcal{H} \subset \mathcal{G}(D)$ of any point $a \in D$ is a maximal compact subgroup in the group $\mathcal{G}(D)$.*

Denote by σ the reflection in the point a. The domain $D \cap \sigma D$ is bounded and invariant with respect to the group $\mathcal{H}$. On the other hand, it is clear that the group $\mathcal{H}$ is closed in the full affine group. Consequently, $\mathcal{H}$ is compact. The maximal compactness of the group $\mathcal{H}$ is proved in the same way as in ([1]), Theorem 2.

Remark. Theorem 1 and its proof remain valid if, instead of the group $\mathcal{G}(D)$, one takes any of its closed subgroups transitive on D.

Theorem 2. *There exists an algebraic affine group $\widetilde{\mathcal{G}}$ such that*

$$\widetilde{\mathcal{G}}_0 \subset \mathcal{G}(D) \subset \widetilde{\mathcal{G}}, \tag{2}$$

where $\widetilde{\mathcal{G}}_0$ is the connected component of the identity of the group $\widetilde{\mathcal{G}}$.

Explanation. An **algebraic affine group** is an affine group singled out from the full affine group by polynomial equations. The full affine group of an n-dimensional space is canonically identified with a subgroup of the full linear group of an $(n+1)$-dimensional space. Namely, to the affine transformation $x \to Ax + a$ of the space R^n there corresponds the linear transformation

$$(x, t) \to (Ax + at, t) \qquad (x \in R^n,\ t \in R^1)$$

of the space $R^{n+1} \simeq R^n \times R^1$. Under this identification, algebraic affine groups correspond to algebraic linear groups, and conversely.

Let G be the Lie algebra of the group $\mathcal{G}(D)$. For any $U \in G$ and $x \in R^n$, by $U(x)$ we shall denote the n-dimensional vector which is the value of the infinitesimal affine transformation U at the point x. The points of the space R^n in whose neighborhood the group $\mathcal{G}(D)$ acts transitively are characterized by the fact that the dimension of the space $\{U(x)\}_{U \in G}$ is equal to n. The points at which it is less than n form an algebraic surface S. The domain D is a connected component of the set $R^n \setminus S$. Let S' be the union of those irreducible components of the surface S which are "sufficiently fully represented" on the boundary Γ of the domain D, namely those whose intersection with Γ is $(n-1)$-dimensional. The closure $\overline{S'}$ of the surface S' contains Γ, and therefore D coincides with a connected component of the set $R^n \setminus \overline{S'}$. Let $\widetilde{\mathcal{G}}$ be the group of affine transformations preserving the algebraic surface S'. It is obvious that $\widetilde{\mathcal{G}}$ is an algebraic affine group and that $\widetilde{\mathcal{G}}_0 \subset \mathcal{G}(D)$. On the other hand, the group $\mathcal{G}(D)$ preserves Γ and, consequently, S', i.e. it is contained in $\widetilde{\mathcal{G}}$.

Definition. An affine Lie group $\mathcal{G}$ is called **triangular** if the principal linear parts of the transformations in $\mathcal{G}$, in some basis, are written as upper triangular matrices.

It is obvious that an affine group is triangular if and only if the linear group identified with it is triangular (see the explanation of Theorem 2).

Theorem 3. *In the group $\mathcal{G}(D)$ there exists a connected triangular subgroup $\mathcal{T}(D)$, transitive on D. All such subgroups are conjugate in $\mathcal{G}(D)$ and act on D simply transitively.*

For the proof it suffices to compare Theorems 1 and 2 and (2), Theorems 1 and 2.

Remark. Theorem 3 and its proof remain valid if, instead of the group $\mathcal{G}(D)$, one takes any of its subgroups that is transitive on D and satisfies condition (2) for some algebraic affine group $\mathcal{G}$.

Corollary. *Every convex homogeneous cone V is spanned by some convex homogeneous domain.*

Let $\mathcal{T}(V)$ be a triangular group, simply transitive on V. Denote by $\mathcal{U}$ the one-parameter group of scalar transformations contained in $\mathcal{G}(V)$. Obviously, $\mathcal{U} \subset \mathcal{T}(V)$. Further, in the space R^n there exists an $(n-1)$-dimensional subspace P, invariant with respect to $\mathcal{T}(V)$. If $\mathcal{T}_1$ is the normal divisor of the group $\mathcal{T}(V)$ consisting of transformations identical in the quotient space R^n/P, then $\mathcal{T}(V) = \mathcal{T}_1\mathcal{U}$. Let D be the section of the cone V by a hyperplane parallel to P. It is easy to see that the cone V is spanned by the convex domain D, and that the group $\mathcal{T}_1$ is transitive on D.

By Theorem 1, in the domain D there exists a measure invariant with respect to $\mathcal{G}(D)$. This measure is unique up to a positive factor. We shall agree to denote its density by $\varphi = \varphi_D$.

Theorem 4. *The quadratic form $d^2 \ln \varphi$ is positive definite at all points of the domain D.*

The domain D is embedded as a plane section in the cone $V(D)$ spanned by it. In this case the function φ_D coincides with the restriction to D of the function $\varphi_{V(D)}$. It remains to apply Theorem 1 from (1).

We shall now establish a one-to-one correspondence between convex homogeneous domains and nonassociative algebras of a special kind.

Choose some point $0 \in D$ and declare it to be the origin of coordinates, thereby introducing in R^n the structure of a vector space. Let T be the Lie algebra of the group $\mathcal{T}(D)$. The mapping $U \to U(0)$ is an isomorphism of the vector spaces T and R^n. For every $a \in R^n$, denote by U_a the element of the Lie algebra T such that $U_a(0) = a$. In the space R^n introduce multiplication by the formula

$$ab = U_a(b) - a. \tag{3}$$

Denote the algebra so obtained by $A(D)$. It is easy to see that the algebra $A(D)$ does not depend on the choice of the group $\mathcal{T}(D)$ or of the point $0 \in D$.

We have

$$U_a(x) = ax + a, \tag{4}$$

whence, by the commutation rule for infinitesimal affine transformations,

$$[U_a, U_b](x) = a(bx) - b(ax) + ab - ba.$$

In particular, $[U_a, U_b](0) = ab - ba$, so that

$$[U_a, U_b] = U_{ab-ba}. \tag{5}$$

This is equivalent to the identity

$$[abc] = [bac] \tag{6}$$

in the algebra A ($[abc] = a(bc) - (ab)c$ is the so-called associator). Algebras satisfying axiom (6) will be called left-symmetric.

In the algebra $A(D)$ put $T_a x = ax$ and $s(a) = \operatorname{Sp} T_a$. We have $\ln \varphi(\exp U_a \cdot 0) = -s(a)$. Taking into account in this equality the terms of the first and second orders of smallness with respect to a, we obtain for the differentials of the function $\ln \varphi$ at the point 0:

$$(d \ln \varphi)_0(a) = -s(a); \qquad (d^2 \ln \varphi)_0(a) = s(a^2). \tag{7}$$

Theorem 4 shows that the quadratic form $s(a^2)$ in the algebra $A(D)$ is positive definite. Further, by (5), $T_{ab-ba} = [T_a, T_b]$, and therefore $s(ab) = s(ba)$. Left-symmetric algebras in which there exists such a linear form s that $s(ab) = s(ba)$ and the quadratic form $s(a^2)$ is positive definite will be called compact. Finally, in the algebra $A(D)$ all eigenvalues of the operators of left multiplication T_a are real. Algebras possessing this property will be called normal. Normal compact left-symmetric algebras will, for short, be called clans.

Theorem 5. *The construction described above, which associates with each convex homogeneous domain D the clan $A(D)$, establishes a one-to-one correspondence between convex homogeneous domains (defined up to equivalence) and clans.*

Let A be any clan. The infinitesimal affine transformations U_a $(a \in A)$, defined in the space A by formula (4), form, by virtue of (5), a Lie algebra. The corresponding affine group acts transitively in some domain D of the space A containing 0. It can be shown that the domain D is convex and that $A \simeq A(D)$, whence theorem 5 follows. We are obliged to omit this proof.

Theorem 5 opens the way to an algebraic study of convex homogeneous domains. Along this path one obtains, in particular:

Theorem 6. *Regarding the domain D as containing 0, denote by C the largest cone contained in D, and let R^k be its linear span. The open kernel V of the cone C in R^k is a homogeneous convex cone, and the domain D consists of those points $(x, u) \in R^k \times R^{n-k} \simeq R^n$ for which*

$$x_0 + x - F(u, u) \in V, \tag{8}$$

where F is a certain bilinear function on $R^{n-k} \times R^{n-k}$ with values in R^k, and x_0 is a certain point of V.

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