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Abstract

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ENTROPY OF THE HOROCYCLE FLOW

(Presented by Academician A. N. Kolmogorov, 3 IX 1960)

§ 1. **Statement of the results.** A. N. Kolmogorov (^{1}) pointed out probabilistic examples of dynamical systems with countably multiple Lebesgue spectrum and a prescribed positive value of the entropy (including $+\infty$). Ya. G. Sinai (^{7}) computed the entropy of the geodesic flow on a compact surface of constant negative curvature and in this way obtained examples of classical dynamical systems with countably multiple Lebesgue spectrum and a prescribed (finite) positive value of the entropy.

Recently I. V. Girsanov constructed an example of a system with countably multiple Lebesgue spectrum and **zero** entropy. Like Kolmogorov' s examples, this example is of probabilistic origin and is not a dynamical system in the classical sense.

In the present note it is proved that *the horocycle flow on a compact surface of constant negative curvature has zero entropy.* From the results of (^{2},^{3}) it follows that the horocycle flow has countably multiple Lebesgue spectrum. Thus we obtain an example of a classical system with countably multiple Lebesgue spectrum and zero entropy.

It should be noted that the system constructed by Girsanov is mixing of all orders. For the horocycle flow the question of multiple mixing remains open.

§ 2. **Description of the horocycle flow** (cf. (^{5})). Let S be the interior of the unit disk with the non-Euclidean metric; G a discrete subgroup of the group of fractional-linear transformations of the complex plane that carry S onto itself; B a fundamental domain of the group G. In the case under consideration this domain is compact. Therefore its boundary does not intersect the unit circle and consists, as is known, of a finite number of segments $l_1, l_2, \dots, l_k$ of geodesics. These segments are pairwise congruent with respect to the group G. Upon identifying congruent points, S becomes a two-dimensional manifold M. The horocycle flow $\{H_t\}$ acts in the space Ω of line elements of the surface M. We shall represent Ω as the collection of line elements whose supports lie in the domain B. Let e be one of these elements, and let $c(e)$ be the horocycle in S determined by it. Suppose that the support of the element e, moving along the horocycle $c(e)$ counterclockwise, first intersects the boundary of the fundamental domain at a point $p \in l_i$. Denote by g_i that transformation from the group G which carries the arc l_i to the congruent arc l_i'. This transformation puts into correspondence with the point p a certain point $p' \in l_i'$, and with the horocycle $c(e)$ a certain horocycle $c'(e)$, intersecting the arc l_i' at the point p'. Under the action of the flow $\{H_t\}$, the support of the line element e moves with unit velocity counterclockwise along the horocycle $c(e)$ to the point p, then jumps to the point p' and moves along the horocycle $c'(e)$ inside the domain B until the intersection of this horocyc-

along the next segment of the boundary, after which there is again a jump similar to the one already described. In this motion the direction of the line element coincides with the outward normal to the corresponding horocycle.

§ 3. **Lemma.** *There exists a partition E of the space Ω into a finite number of nonintersecting measurable sets $E_1, E_2, \dots, E_\nu$, and a positive δ, such that for every positive $\tau < \delta$*

$$\prod_{n=0}^{\infty} H_{n\tau} E = \varepsilon \bmod 0^*. \tag{1}$$

Proof. Since two geodesics cannot intersect inside the unit circle at a zero angle, all the angles between segments of the boundary of the domain B are positive. Using this, one can show that there exists an N, depending only on δ, such that any segment of a horocycle of length δ intersects no more than N domains congruent to the domain B (such domains do not intersect and fill the interior of the unit circle). Therefore, under the motion described above under the action of the flow $\{H_t\}$, any line element makes no more than N jumps in time δ. Let $A(i_1, i_2, \dots, i_N)$ be the set of line elements with carriers in the domain B which, in time τ, successively make jumps across segments with numbers $i_1, i_2, \dots, i_N$ of the boundary of the domain B; if in time τ there are $m < N$ jumps, then we put $i_{m+1} = i_{m+2} = \dots = i_N = 0$. Denote by A_1 the finite partition of the space Ω formed by the sets $A(i_1, i_2, \dots, i_N)$. Let K be some circle lying in the domain B, and L some Euclidean straight line passing through the Euclidean center O of this circle. The set of line elements whose carriers lie on one particular side of the line L shall be denoted by A', and the complement of this set in the whole space Ω by A''. The partition of the space Ω into two parts obtained in this way shall be denoted by A_2. Let e_0 be the line element with carrier at the point O, directed along the line L; $c(e_0)$ the horocycle in S determined by this element; p_1 and p_2 the points of intersection of $c(e_0)$ with the boundary of the circle K; ρ_1 and ρ_2 the distances, measured along the horocycle $c(e_0)$, from the point O to the points p_1 and p_2, respectively. Divide the domain B into a finite number of domains in each of which the distance between any two points, measured along the horocycle joining them, does not exceed $\frac{1}{2}\min(\rho_1, \rho_2)$; denote this partition by A_3, and we shall show that the partition $E = A_1 \cdot A_2 \cdot A_3$ and the number $\delta = \frac{1}{2}\min(\rho_1, \rho_2)$ are the desired ones.

Only relation (1) needs verification. Let the line elements e_1 and e_2 determine on the surface M the horocycles $\bar{c}(e_1), \bar{c}(e_2)$, and suppose that the line elements $H_{n\tau}e_1, H_{n\tau}e_2$ belong, for every n $(0 \le n < \infty)$, to one and the same set of the partition E. First suppose that $e_1 \neq e_2$, $\bar{c}(e_1) = \bar{c}(e_2)$, and moreover $e_1 = H_\gamma e_2$, where $\gamma > 0$. Then for all $n > 0$

$$H_{n\tau} e_1 = H_{n\tau+\gamma} e_2. \tag{2}$$

Since the spectrum of the flow $\{H_t\}$ is continuous, H_τ is an ergodic automorphism, and one may assume that the set of line elements of the form $H_{n\tau} e_1$, $n > 0$, is everywhere dense in the space Ω. It follows that for every neighborhood of the element e_0 there is an integer $r > 0$ such that the line element $H_{r\tau} e_1$ falls into the intersection of the set A' with this neighborhood. If the neighborhood of the element e_0 is chosen sufficiently small, and if by A' we denote that element of the partition A_2 to which the line elements $H_t e_0$ belong for small positive t, then, by equality (2), $H_{r\tau} e_2 \in A''$. Since this inclusion contradicts our assumption concerning the line elements $H_{n\tau} e_1$ and $H_{n\tau} e_2$ for $n > 0$, in the case under consideration $e_1 = e_2$.

Let now $\bar{c}(e_1) \neq \bar{c}(e_2)$. From the fact that $H_{n\tau} e_1$ and $H_{n\tau} e_2$, for every $n \geq 0$, belong to one and the same element of the partition A_1, it follows that the line elements e_1 and e_2, moving uniformly along the horocycles $c(e_1)$ and $c(e_2)$ in the space S, are at each of the instants $n\tau$ $(n = 1, 2, ...)$ in one and the same domain, congruent to the domain B, and, consequently, the distance between their carriers at the instants $n\tau$ does not exceed the diameter of the domain B. However, it is not difficult to verify that even in the case when the horocycles $c(e_1)$ and $c(e_2)$ have a common infinitely remote point, the distance between points moving along them with unit speed tends to infinity (it grows with speed of order $\log t$). The contradiction obtained completes the proof of the lemma.

§ 4. **Entropy of the flow $\{H_t\}$.** As shown in (^{4}), for the entropy $h(S_t)$ of an automorphism belonging to a measurable flow $\{S_t\}$, the formula

$$h(S_t) = |t| h(S_1) \tag{3}$$

holds. The entropy of the flow $\{S_t\}$ is defined as $h(S_1)$.

Comparison of formula (3) and Theorem 2 from (^{6}) with the lemma proved above immediately leads to the conclusion that the entropy of the flow $\{H_t\}$ is equal to zero.

Remark. The computation of the entropy of the flow $\{H_t\}$ reveals for the first time an essential difference between the horocycle flow and the geodesic flow. In particular, it shows that the horocycle flow is not a Kolmogorov flow (cf. (^{7})). An attempt to transfer to the geodesic flow $\{\Gamma_t\}$ the construction carried out in the proof of the lemma leads to a finite partition E', for which, instead of (1),

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* ε mod 0 means the partition of the space into individual points, up to a set of measure 0.

the relation

$$\prod_{n=-\infty}^{\infty} \Gamma_{n\tau} E' = \varepsilon \bmod 0, \tag{4}$$

holds, if τ is positive and does not exceed some δ'. Equality (4) is also of interest. It shows that there exists a finite partition of the space Ω which is a generator (^{8}) for every automorphism Γ_τ with sufficiently small τ.

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