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Abstract

Full Text

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ON SYMMETRIC SPACES OF AFFINE CONNECTION OF THE FIRST CLASS

(Presented by Academician P. S. Aleksandrov on 24 IV 1961)

In an N -dimensional affine space R_N , consider an n -dimensional surface. To each point M of this surface attach an n -dimensional plane S (the base plane) passing through this point, and an $(N - n)$ -dimensional plane Σ (the supporting plane), having with S only the single common point M . In the manifold of base planes S , an affine connection is established by projecting a neighboring base plane onto the original one parallel to the original supporting plane Σ . The problem of immersing an n -dimensional space of affine connection A_n in R_N consists in finding in R_N an n -dimensional surface, equipped with base and supporting planes, on which the connection established by projection would coincide with the connection in A_n . In this formulation the immersion problem was considered in the works (1-5).

Let R_N be an affine space of minimal dimension N into which the given n -dimensional A_n can be immersed. The difference $N - n$ is called the **class** of A_n . In the present paper we consider symmetric A_n of the 1st class with nondegenerate Ricci tensor. In this case the base planes are hyperplanes, and the supporting planes are lines.

1. In the subsequent arguments we shall have to use the following lemma, which we state here without proof.

Lemma. An n -dimensional space of affine connection A_n without torsion, of the 1st class, with nondegenerate Ricci tensor R_{ij} ($\text{Det} \|R_{ij}\| \neq 0$), can be realized in R_{n+1} only on a hypersurface of rank n , under the condition that the tangent hyperplane is chosen as the base hyperplane. In all other realizations the tensor R_{ij} is degenerate; and if a non-tangent hyperplane is chosen as the base hyperplane, then, in addition, the matrix composed of the components of the curvature tensor

$$\|R^i_{jkl}\| \tag{1}$$

where i denotes rows and j, k, l columns, must necessarily have rank 1.

In view of the lemma, we must refer the hypersurface realizing our connection to a frame of the 1st order. By its equipment we shall henceforth understand only the equipment by supporting lines.

2. Consider a symmetric A_n ($n > 2$) of the 1st class, of which, in addition to nondegeneracy of the Ricci tensor R_{ij} , we require that the matrix (1) have rank > 2 . The realization equations for A_n in R_{n+1} have the form

$$\begin{aligned} \omega^{n+1} = 0; \quad \omega_i^{n+1} = \Lambda_{ij}\omega^j; \quad \omega_{n+1}^i = \Lambda_j^i\omega^j \\ (\Lambda_{ij} = \Lambda_{ji}; \quad \text{Det } \|\Lambda_{ij}\| \neq 0; \quad i, j = 1, \dots, n). \end{aligned} \quad (2)$$

The components of the curvature tensor of our A_n are expressed through Λ_{ij} and Λ_j^i in the following way:

$$R^i_{jkl} = \frac{1}{2} (\Lambda_{jk}\Lambda_l^i - \Lambda_{jl}\Lambda_k^i). \quad (3)$$

Since $\text{Det } \|\Lambda_{ij}\| \neq 0$, we may introduce into consideration the object Λ^{ij} , whose components are the elements of the matrix $\|\Lambda^{ij}\|$ inverse to the matrix $\|\Lambda_{ij}\|$. From expressions (3) there follow the relations:

$$\Lambda^{ls}\Lambda_{jkl}^i = 0 \quad (s \neq k, l); \quad (4)$$

$$\Lambda^{jk}\Lambda_{ikl}^i = \frac{1}{2}\Lambda_l^i \quad (\text{no summation over } k). \quad (5)$$

Covariant differentiation of relations (4) with respect to the connection of the space A_n gives, by virtue of (5):

$$\Lambda_l^i\Lambda^{as}\Lambda_{kam} - \Lambda_k^i\Lambda^{as}\Lambda_{lam} = 0 \quad (s \neq k, l), \quad (6)$$

where Λ_{ijk} are the components of the fundamental object of the 3rd order of the hypersurface.

From the fact that the matrix (1) has rank > 2 , it follows that the matrix $\|\Lambda_j^i\|$ also has rank > 2 . In view of this, from relations (6) we obtain

$$\Lambda^{as}\Lambda_{kam} = 0 \quad (s \neq k). \quad (7)$$

Since Λ_{ijk} are symmetric in all their indices, it follows from equations (7) that

$$\Lambda_{ijk} = 0. \quad (8)$$

The equalities (8) can hold if and only if our hypersurface is a hyperquadric and is furnished with Blaschke affine normals.

Thus, we have proved the following theorem.

Theorem 1. *Symmetric A_n ($n > 2$) of the first class with nondegenerate Ricci tensor, for which the matrix (1) has rank > 2 , can be realized in R_{n+1} only on hyperquadrics of rank n , furnished with Blaschke affine normals.*

- From the equations obtained by G. F. Laptev (6) for a hyperquadric in projective space, it follows that the equations of a hyperquadric of rank n , furnished with Blaschke affine normals, have the following form:

$$\omega^{n+1} = 0; \quad \omega_i^{n+1} = \Lambda_{ij}\omega^j; \quad \omega_{n+1}^i = -\frac{\hat{l}}{n+2}\omega^i, \quad (9)$$

$$d\Lambda_{ij} = \Lambda_{ik}\omega_j^k + \Lambda_{kj}\omega_i^k - \Lambda_{ij}\omega_{n+1}^k; \quad d\hat{l} = \hat{l}\omega_{n+1}^k \quad (\text{Det } \|\Lambda_{ij}\| \neq 0).$$

The components of the curvature tensor of the affine connection induced on the hyperquadric by such an equipment, and the components of the Ricci tensor of this connection, have the form:

$$R^i{}_{jkl} = \frac{\hat{l}}{2(n+2)} (\delta_k^i \Lambda_{jl} - \delta_l^i \Lambda_{jk}); \quad (10)$$

$$R_{ij} = -\frac{\hat{l}(n-1)}{2(n+2)} \Lambda_{ij}. \quad (11)$$

It is easy to see that this connection is symmetric and equiaffine.

The curvature tensor satisfies the relations

$$R^i{}_{jkl} = -\frac{1}{n-1} (\delta_k^i R_{jl} - \delta_l^i R_{jk}). \quad (12)$$

If we introduce into consideration the metric tensor

$$g_{ij} = -\frac{n}{R} R_{ij}, \quad \text{where } R = \text{const}, \quad (13)$$

then relations (12) take the form

$$R^i{}_{jkl} = K (\delta_k^i g_{jl} - \delta_l^i g_{jk}), \quad \text{where } K = \frac{1}{n(n-1)} R. \quad (14)$$

Consequently, our connection may be regarded as the connection of a Riemannian space of constant curvature.

This allows us to state the following theorem:

Theorem 2. *The affine connection induced in R_{n+1} on a hyperquadric of rank n by Blaschke affine normals is the connection of a Riemannian space of constant curvature.*

4. P. A. Shirokov proved that Riemannian spaces of constant curvature are spaces of the 1st class and that there exist no symmetric Riemannian spaces of the 1st class with nondegenerate Ricci tensor, distinct from spaces of constant magnitude. From Theorems 1 and 2 it follows that there exist no non-Riemannian symmetric spaces of the 1st class satisfying the restrictions indicated in Theorem 1. We obtain the following theorem:

Theorem 3. *There exist no symmetric spaces of affine connection A_n ($n > 2$) of the 1st class with nondegenerate Ricci tensor, for which the matrix (1) has rank > 2 , distinct from Riemannian spaces of constant curvature.*

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Note: Figure translations are in progress. See original paper for figures.

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