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Abstract

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MATHEMATICS

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ON A LATTICE-THEORETICAL THEOREM OF THE TYPE OF GRUSHKO' S THEOREM

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For free decompositions of groups with a finite number of generators there is Grushko' s theorem ([1,2]), which can be formulated in the following form:

If a free group S with a finite number of generators is mapped homomorphically onto a group G decomposed into a free product of subgroups $A_1, A_2, \dots, A_k$, then in S one can choose such a system of free generators that, under the homomorphism considered, each of these generators is mapped into one of the free factors $A_1, A_2, \dots, A_k$.

A similar theorem has also been proved for nonassociative algebras ([3]). In lattice theory an analogous theorem is false, since already under a homomorphic mapping of the free lattice with three generators $FL(3)$ onto $FL(2)$ ([9]) there is no system of generating elements of the lattice $FL(3)$ each of whose elements would be mapped into one of the free factors of the lattice

$$FL(2) = FL(1) * FL(1).$$

However, in lattice theory there is a theorem analogous to one of the main corollaries of Grushko' s theorem (see ([1]), and also ([7])), namely

Theorem. *If*

$$L = L_1 * L_2 * \dots * L_k \tag{1}$$

is an arbitrary decomposition into a free product ([4]) of a lattice L with a finite number of generators, then the minimal number of generating elements of the lattice L is equal to the sum of the corresponding numbers for the free factors $L_1, L_2, \dots, L_k$.

Below the idea of the proof of this theorem is set forth.

It is clear that in proving it one may restrict oneself only to the case when in the decomposition (1) $k = 2$, i.e.

$$L = L_1 * L_2. \tag{2}$$

The elements of the lattices L_1 and L_2 will in what follows be denoted respectively by the letters x and y with subscripts.

An element $u \in L$ will be called **inner with respect to the lattice** L_1 if there exist two elements x_p, x_q such that

$$x_p \leqslant u \leqslant x_q.$$

An element inner with respect to the lattice L_2 is defined analogously.

An element $u \in L$ will be called **inner** if it is inner with respect to at least one of the lattices L_1, L_2.

It is easy to see that the elements of the lattices L_1 and L_2 are inner and that one element cannot be inner with respect to both lattices. The latter follows from the fact that no two elements x_{p_1}, y_{q_1} are comparable ([4]).

It follows from ([4]) that the free product of the lattices L_1 and L_2 coincides with the free extension $FL(P)$ of the latticeoid P, which is the cardinal sum* $L_1 + L_2$ of the lattices L_1 and L_2.

In ([6]) it is shown that the lattice $FL(P)$ may be regarded as the set-theoretic union of a sequence of extensions of the latticeoid P:

$$P = P_0 \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \dots \Rightarrow P_{\alpha-1} \Rightarrow P_\alpha \Rightarrow \dots, \tag{3}$$

in which each of the latticeoids is identically embedded in the next one.

If α is a regular number, then the latticeoid P_α is obtained from $P_{\alpha-1}$ by adding one new element z and the relation

$$z_s * z_t = z, \tag{4}$$

where $z_s \neq z$, $z_t \neq z$, and $z_s * z_t$ is not defined in $P_{\alpha-1}$, and by reducing to closed form the system of generating elements consisting of the elements of the latticeoid $P_{\alpha-1}$ and the element z, and the system of defining relations consisting of all the relations of Klein' s tables for joins and meets of the elements of the latticeoid $P_{\alpha-1}$ and relation (4). This process in the general case was described by Evans ([8]). In the present case it will consist of all possible substitutions of elements of the latticeoid $P_{\alpha-1}$ and of the element z into the axioms of a lattice ([5]). Some substitutions will give relations between elements that hold in $P_{\alpha-1}$, while other substitutions will give new relations connecting the element z with elements of the latticeoid $P_{\alpha-1}$.

If α is a limit number, then $P_\alpha = \bigcup_{\mu<\alpha} P_\mu$ ([6]). Therefore, in what follows we shall regard the number α as regular.

The minimal number of generating elements of the latticeoid P_α will be called its rank and denoted by $\operatorname{rang} P_\alpha$.

Relation (4) may have the form

$$z_s \cup z_t = z \quad \text{and} \quad z_s \cap z_t = z.$$

By the principle of duality it is enough to consider only one of these cases.

In the following lemmas we shall assume that the latticeoid $P_{\alpha-1}$ is obtained from P_α by adding the element z with the relation

$$z_s \cup z_t = z. \tag{5}$$

Lemma 1. *In the latticeoid P_α the element z does not decompose into the meet of two distinct elements different from z, i.e., if in P_α there is a relation of the form*

$$u_1 \cap u_2 = z, \tag{6}$$

then either $u_1 = z$ or $u_2 = z$.

As noted above, all relations containing z, except relation (5), are obtained as the result of substitutions of elements into the axioms. Consequently, all relations containing z may be renumbered in the order in which they are obtained. After this, the assertion of Lemma 1 is easily proved by induction on the number of the relation.

Here it is useful to take into account that a relation of the form (6) with $u_1 \neq z$ and $u_2 \neq z$, up to commutativity, can be obtained only by substituting elements into the axiom

$$a \cap (b \cap c) = (a \cap b) \cap c.$$

Similarly one proves:

Lemma 2. *If in P_α there is a relation*

$$z \cap u = v, \quad u \neq v \neq z, \tag{7}$$

* For the definition of the cardinal sum, see ([10]).

then in P_α there exists a set of elements $w_1, w'_2, \dots, w'_l$ such that:
1) $w'_i > z,\ i = 1, 2, \dots, l$; 2) $\prod_{i=1}^{l} w'_i \cap u = v$.

Lemma 3. If $u \in P_{\alpha-1}$ and in P_α

$$u = \varphi(v_1, v_2, \dots, v_m, z), \qquad v_i \neq z,$$

then

$$u = \psi(v_1, v_2, \dots, v_m, w_1, w_2, \dots),$$

where $w_1, w_2, \dots$ is the set W of all elements covering z in P_α.

Lemma 3, on the basis of Lemma 2, can be proved by induction on the rank of the word φ.

From Lemma 3 it follows immediately:

Lemma 4. If $T = \{u_1, u_2, \dots, u_r, z\}$ is a system of generating elements of the structuroid P_α, then

$$T_1 = \{u_1, u_2, \dots, u_r, w_1, w_2, \dots\}, \qquad w_i \in W,$$

is also a system of its generators.

Lemma 5. If an element u belongs to an irreducible system of generating elements of a structure L, then it is an inner element.

Every irreducible system of generating elements of a structure L belongs to some structuroid of the series (3), and therefore the proof of Lemma 5 can be carried out by induction on the number of the structuroid in the series (3). In doing so, the induction hypothesis permits one to reduce it to the proof only of the following assertion:

If $T = \{u_1, u_2, \dots, u_r, z\}$ is an irreducible system of generating elements of the structuroid P_α, then the element z is inner.

The validity of this last assertion follows from the fact that at least one of the elements z_s, z_t and one of the elements of the set W are inner, and moreover with respect to one and the same structure.

In an analogous way the following is proved:

Lemma 6. In none of the structuroids of the series (3) does there exist a quadruple of inner elements u', u'', v', v'' satisfying the conditions:
1) $u' \neq u'',\ v' \neq v''$; 2) u' covers v'', u'' covers v'; 3) $u' > v'$, $u'' > v''$.

Lemma 7. $\operatorname{rang} L = \operatorname{rang} P$.

For the proof of Lemma 7 it is enough to prove that

$$\operatorname{rang} P_\alpha = \operatorname{rang} P_{\alpha-1}. \qquad (8)$$

Since every system of generating elements of the structuroid $P_{\alpha-1}$ is a system of generating elements of the structuroid P_α, it follows that

$$\operatorname{rang} P_\alpha \leqslant \operatorname{rang} P_{\alpha-1}. \qquad (9)$$

We shall show that the rank of the structuroid P_α cannot be less than the rank of the structuroid $P_{\alpha-1}$. For this it is sufficient to show that if T is an irreducible system of generating elements of the structuroid P_α, then there exists a system of generating elements of the structuroid $P_{\alpha-1}$ containing no more than $r+1$ elements.

Since T is a system of generating elements of the structuroid P_α and $z = z_s \cup z_t$, it follows that

$$T' = \{u_1, u_2, \dots, u_r, z_s, z_t\}$$

is a system of generating elements of the structuroid $P_{\alpha-1}$. If T' is reducible, then the assertion is true. If, however, it is irreducible, then, by Lemma 5, z_s, z_t are inner elements.

Moreover, by Lemma 4, T_1 is a system of generating elements of the structuroid $P_{\alpha-1}$.

Select from T_1 an irreducible system of generators T_2. If T_2 contains two elements from W, then in the structuroid $P_{\alpha-1}$ there will be a quadruple of elements satisfying the conditions of Lemma 6, which is impossible. Consequently, T_2 contains only one element from W, and since $T_2 \subseteq P_{\alpha-1}$, we have

$$\operatorname{rang} P_{\alpha-1} \leqslant \operatorname{rang} P_\alpha,$$

which, together with relation (9), gives

$$\operatorname{rang} P_\alpha = \operatorname{rang} P_{\alpha-1}.$$

Lemma 7 says that the minimum number of generating elements of the lattice L is equal to the minimum number of generating elements of the structuroid P, which proves the theorem formulated at the beginning.

We indicate some consequences of the theorem proved, analogous to the consequences that follow from the group-theoretic theorem of Grushko ([1]).

1. If L is a lattice with a finite number of generating elements, then every element of any of the minimal* systems of generators is contained in one of the factors of any of its free decompositions.

2. Every free decomposition of a lattice with n generators consists of no more than n factors, and therefore every lattice with a finite number of generators can be decomposed into a free product of a finite number of indecomposable lattices.

3. If a lattice L with n generators decomposes into a free product of n lattices, then L is a free lattice.

4. A free lattice of rank n cannot possess a system of generators consisting of fewer than n elements.

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* The concept of a minimal system of generators is defined in the same way as in group theory ([1]).

Note: Figure translations are in progress. See original paper for figures.