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Abstract

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HYDROMECHANICS

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ON THE STATISTICAL DYNAMICS OF AN INCOMPRESSIBLE TURBULENT FLUID

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In the modern theory of turbulence, a fluid flow is usually described by two-point statistical moments, i.e., correlations of hydrodynamic quantities referring to two points separated by a finite distance. The complexity of such a description has compelled one to confine oneself mainly to the simplest case of homogeneous turbulence, but even under such a restriction only incomplete successes have been achieved. It can hardly be hoped, therefore, that a theory of inhomogeneous turbulence operating with two-point moments could make it possible to carry concrete problems through to completion. This has led us to return to one-point moments and to try to obtain for them a closed system of differential equations.

In the author’ s work (^{1}), a turbulent flow was described by the following quantities: the mean velocity $U_i = \overline{u_i}$ at each given point (and at a given instant of time), then the second and third moments of the fluctuating velocities $v_i = u_i - U_i$, $R_{ij} = \overline{v_i v_j}$ and $S_{ijk} = \overline{v_i v_j v_k}$, and, finally, the dissipation of energy at the given point $Q = \nu\overline{(dv_i/dx_k)^2}$, where ν is the viscosity.

In deriving statistical differential equations for these basic quantities from the Navier–Stokes equation, other moments also appear in the equations; these we expressed approximately in terms of the enumerated basic ones. The fourth moments of the fluctuating velocities are then expressed in terms of the second moments in a known way. As for the moments that include the pressure fluctuation $q = p - P$ (where $P = \overline{p}$), they are expressed through the basic moments on the basis of a close analogy with the corresponding terms in the Boltzmann kinetic equation for ideal gases.

For U_i, R_{ij}, and S_{ijk}, the following differential equations were obtained in (^{1}) in this way:

$$dU_i/dt + \partial R_{ik}/\partial x_k + \partial P/\partial x_i = \nu\, \partial^2 U_i/\partial x_k^2, \qquad \partial U_k/\partial x_k = 0; \tag{1}$$

$$dR_{ij}/dt + R_{ik}\partial U_j/\partial x_k + R_{jk}\partial U_i/\partial x_k + \partial S_{ijk}/\partial x_k + \beta Q\,(R_{ij} - {}^1/_3\, \delta_{ij}R)/R +$$

$$+B_{ij} - {}^1/_3\delta_{ij}B_{kk} + {}^2/_3\delta_{ij}Q = \nu\, \partial^2 R_{ij}/\partial x_k^2; \tag{2}$$

$$dS_{ijk}/dt + S_{ijl}\partial U_k/\partial x_l + S_{jkl}\partial U_i/\partial x_l + S_{ikl}\partial U_j/\partial x_l +$$

$$+R_{il}\partial R_{jk}/\partial x_l + R_{jl}\partial R_{ik}/\partial x_l + R_{kl}\partial R_{ij}/\partial x_l + \beta_1 Q S_{ijk}/R = 0. \tag{3}$$

The trace of equation (2) gives the energy equation

$${}^1/_2\, dR/dt + R_{ik}\partial U_i/\partial x_k + {}^1/_2\, \partial S_k/\partial x_k + Q = {}^1/_2 \nu\, \partial^2 R/\partial x_k^2, \tag{2a}$$

where $S_k = S_{kll}$ is twice the turbulent energy flux.

The tensor B_{ij} appears in equation (2) in a skewed flow; it has significance mainly near walls. In (^{2}) the following was obtained for it···

expression

$$B_{ij} = \frac{1}{2} N \frac{\partial U_k}{\partial x_i}\frac{\partial U_k}{\partial x_j}\left(\beta - 2 + 3\beta^2 N \frac{R_{lm}}{R^2}\frac{\partial U_l}{\partial x_m}\right). \tag{4}$$

Here N denotes the turbulent viscosity

$$N = -R_{ik}\frac{\partial U_i}{\partial x_k}\left(\frac{\partial U_l}{\partial x_m}\right)^{-2}. \tag{5}$$

The equation for Q obtained in (^{2}) cannot be considered satisfactory either in its derivation or, most importantly, because it lacks terms with second derivatives of Q, which undoubtedly must enter.

Now, along with the dissipation Q, we shall also consider the "flux of dissipation"

$$C_i = \overline{v_i(\partial v_k/\partial x_l)^2}. \tag{6}$$

We shall derive differential equations for Q and C_i similarly to the way in which in (^{1}) the equations for R_{ij} and S_{ijk} were obtained.

From the Navier–Stokes equation for Q one obtains:

$$\frac{dQ}{dt}+2\nu\,\frac{\partial U_i}{\partial x_k}\left(\overline{\frac{\partial v_i}{\partial x_l}\frac{\partial v_k}{\partial x_l}}+\overline{\frac{\partial v_l}{\partial x_i}\frac{\partial v_l}{\partial x_k}}\right)+2\nu\,\frac{\partial^2 U_i}{\partial x_k\partial x_l}\,\overline{v_k\,\frac{\partial v_i}{\partial x_l}}+\frac{\partial C_k}{\partial x_k}+2\nu\,\overline{\frac{\partial v_i}{\partial x_k}\frac{\partial v_i}{\partial x_l}\frac{\partial v_k}{\partial x_l}}+2\nu\,\frac{\partial}{\partial x_k}\,\overline{\frac{\partial q}{\partial x_l}\frac{\partial v_k}{\partial x_l}}+2\nu^2\,\Bigg(\overline{\frac{\partial^2 v_i}{\partial x_k\partial x_l}} \tag{7}$$

In this equation we must isolate the principal terms and neglect the less essential ones. In the second term of the equation, the components of the tensor

$$\overline{\frac{\partial v_i}{\partial x_k}\frac{\partial v_j}{\partial x_l}}$$

are determined mainly by small-scale turbulence, whose anisotropy is small. However, if here, similarly to what we did in (1) in deriving the equation for R_{ij}, one puts

$$\nu\,\overline{\frac{\partial v_i}{\partial x_k}\frac{\partial v_j}{\partial x_l}}=\frac{\delta_{ij}\delta_{kl}}{9}\,Q, \tag{8}$$

then the second term of equation (7) generally turns to zero, since $\partial U_k/\partial x_k = 0$. Therefore here it is necessary to take into account the anisotropy of this tensor, which in the equation for R_{ij} was inessential. As the simplest assumption, we shall take

$$2\nu\left(\overline{\frac{\partial v_i}{\partial x_l}\frac{\partial v_k}{\partial x_l}}+\overline{\frac{\partial v_l}{\partial x_i}\frac{\partial v_l}{\partial x_k}}\right)=\alpha\,\frac{Q}{R}\,R_{ik}, \tag{9}$$

where α is a dimensionless coefficient that must be determined from experiment.

The third and fifth terms can apparently be neglected. We shall also neglect the sixth term, just as in deriving the equation for R_{ij} in (1) we neglected analogous terms with qv_i, associated with turbulent transfer of energy by pressure pulsations, i.e., longitudinal waves. For the seventh term, expressing it through our basic quantities, we put:

$$2\nu^2\left(\overline{\partial^2 v_i/\partial x_k\partial x_l}\right)^2=\gamma Q^2/R, \tag{10}$$

where γ is again an empirical dimensionless coefficient. The value of this coefficient is determined if one takes into account the known law of decay of the intensity of homogeneous turbulence with time at the initial stage of degeneration (3), $R \sim 1/t$. Indeed, the energy equation (2a) and equation (7) give, for the homogeneous case,

$$\frac{1}{2}\,dR/dt + Q = 0, \qquad dQ/dt + \gamma Q^2/R = 0. \tag{11}$$

In order that this yield $R \sim 1/t$, it must be that $\gamma = 4$.

Thus we obtain the desired equation for Q

$$\frac{dQ}{dt} + \frac{\partial C_k}{\partial x_k} + \alpha \frac{Q}{R} R_{ik} \frac{\partial U_i}{\partial x_k} + 4\frac{Q^2}{R} = \nu \frac{\partial^2 Q}{\partial x_k^2}. \tag{12}$$

We pass to the equation for the dissipation flux C_i. From the Navier–Stokes equation one obtains

$$\begin{aligned}
&\frac{dC_i}{dt} + 2\nu \frac{\partial U_k}{\partial x_l}\left(\overline{v_i \frac{\partial v_k}{\partial x_m}\frac{\partial v_l}{\partial x_m}} + \overline{v_i \frac{\partial v_m}{\partial x_k}\frac{\partial v_m}{\partial x_l}}\right) + 2\nu \frac{\partial^2 U_m}{\partial x_k \partial x_l}\overline{v_i v_k \frac{\partial v_m}{\partial x_l}} \\
&\quad + C_k \frac{\partial U_i}{\partial x_k} + 2\nu \overline{v_i \frac{\partial v_k}{\partial x_l}\frac{\partial^2 v_k v_m}{\partial x_l \partial x_m}} - 2\nu \overline{v_i \frac{\partial v_k}{\partial x_l}\frac{\partial^2 v_k v_m}{\partial x_l \partial x_m}} \\
&\quad + \nu \overline{\frac{\partial v_i v_k}{\partial x_k}\left(\frac{\partial v_l}{\partial x_m}\right)^2} - \nu \frac{\partial \overline{v_i v_k}}{\partial x_k}\overline{\left(\frac{\partial v_l}{\partial x_m}\right)^2} + 2\nu \overline{\frac{\partial^2 q}{\partial x_k \partial x_l} v_i \frac{\partial v_k}{\partial x_l}} + \nu \frac{\partial q}{\partial x_i}\left(\frac{\partial v_k}{\partial x_l}\right)^2 \\
&\quad + 2\nu^2 v_i \left(\overline{\frac{\partial^2 v_k}{\partial x_l \partial x_m}}\right)^2 + 2\nu^2 \overline{\frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial x_k}\left(\frac{\partial v_l}{\partial x_m}\right)^2} = \nu \frac{\partial^2 C_i}{\partial x_k^2}.
\end{aligned} \tag{13}$$

Here again we must separate out the principal terms. The second term can undoubtedly be neglected in comparison with the fourth. We shall also neglect the third term–it is analogous to the third term in equation (7) for Q. The fourth moments entering the fifth and seventh terms will be expressed approximately through the corresponding second moments, just as this was done for the fourth moments of v_i in the equation for S_{ijk}, assuming thereby that at each point, in the first approximation, there is not only a normal distribution of the velocity fluctuations, but also a normal distribution of their derivatives. Then

$$\begin{aligned}
\overline{v_i \frac{\partial v_k}{\partial x_l}\frac{\partial^2 v_k v_m}{\partial x_l \partial x_m}} &= \overline{v_i \frac{\partial v_k}{\partial x_l}}\;\overline{\frac{\partial^2 v_k v_m}{\partial x_l \partial x_m}} + \overline{\frac{\partial v_k}{\partial x_l}\frac{\partial v_m}{\partial x_l}}\;\overline{v_i \frac{\partial v_m}{\partial x_m}} \\
&\quad + \overline{v_i \frac{\partial v_m}{\partial x_l}}\;\overline{\frac{\partial v_k}{\partial x_l}\frac{\partial v_k}{\partial x_m}} + \overline{v_i v_m}\;\overline{\frac{\partial v_k}{\partial x_l}\frac{\partial^2 v_k}{\partial x_l \partial x_m}} + \overline{v_i \frac{\partial^2 v_k}{\partial x_l \partial x_m}}\;\overline{v_m \frac{\partial v_k}{\partial x_l}}.
\end{aligned}$$

The first of these terms cancels with the sixth term in (13). Of the remaining terms, the principal one is the penultimate, to which we shall restrict ourselves. It is equal to $\frac{1}{2} R_{ik} \partial Q/\partial x_k$ (the second and third terms are small, since they vanish if the small-scale turbulence is assumed isotropic).

We transform the seventh term similarly:

$$\overline{\frac{\partial v_i v_k}{\partial x_k}\left(\frac{\partial v_l}{\partial x_m}\right)^2} = \frac{\partial \overline{v_i v_k}}{\partial x_k}\overline{\left(\frac{\partial v_l}{\partial x_m}\right)^2} + 2\overline{\frac{\partial v_i}{\partial x_k}\frac{\partial v_l}{\partial x_m}}\,\overline{v_k\frac{\partial v_l}{\partial x_m}}.$$

The first of these two terms cancels with the eighth term in (13). The second gives, if condition (8) is adopted here, $\frac{2}{9}Q\,\partial R_{ik}/\partial x_k$. The ninth and tenth terms, which contain the pressure fluctuations q, we express through the principal quantities, just as was done in (^{1}) with the analogous terms in deriving equation (3) for S_{ijk}. We put

$$2\nu\overline{\frac{\partial^2 q}{\partial x_l \partial x_m} v_i \frac{\partial v_l}{\partial x_m}} + \nu\overline{\frac{\partial q}{\partial x_i}\left(\frac{\partial v_l}{\partial x_m}\right)^2} = \frac{\beta_2 Q}{R} C_i \tag{14}$$

with dimensionless coefficient β_2. In comparison with this expression the terms proportional to ν^2 in (13) may be neglected. As a result, the equation for C_i is obtained:

$$\frac{dC_i}{dt} + C_k\frac{\partial U_i}{\partial x_k} + R_{ik}\frac{\partial Q}{\partial x_k} + \frac{2}{9}Q\frac{\partial R_{ik}}{\partial x_k} + \frac{\beta_2 Q}{R}C_i = 0. \tag{15}$$

Equations (1), (2), (3), (12), and (15) give a complete system of differential equations for the quantities U_i, R_{ij}, S_{ijk}, Q, and C_i. In general, 23 quasilinear equations are obtained for the components of these quantities. In various concrete cases the number of equations must, of course, be reduced owing to the symmetry of the problem. In addition, in stationary cases ($d/dt = 0$) some of the equations reduce to purely algebraic relations among the quantities under consideration.

Let us consider a turbulent boundary layer. Approximately, it may always be regarded as plane; we direct the x-axis along the flow, and y normal to the wall. In the first approximation we then have (^{2})

$$R_{xy} = \text{const}, \qquad \partial S/\partial y = 0. \tag{16}$$

The energy equation (2a) and the component of equation (2) for R_{xy} give, if their right-hand sides are neglected, which in the region of developed turbulence are unimportant:

$$Q = -R_{xy}\partial U_x/\partial y, \qquad R_{yy} = \beta R_{xy}^2/R. \tag{17}$$

According to equation (15), in this case

$$C_y = -\frac{RR_{yy}}{\beta_2 Q}\frac{\partial Q}{\partial y} = -\frac{\beta}{\beta_2}\frac{R_{xy}^2}{Q}\frac{\partial Q}{\partial y}. \tag{18}$$

Substituting this value into equation (12) for Q, we obtain, again discarding the right-hand side of the equation:

$$-\frac{\beta}{\beta_2}\frac{\partial}{\partial y}\left(\frac{R_{xy}^2}{Q}\frac{\partial Q}{\partial y}\right) + (4-\alpha)\frac{Q^2}{R} = 0. \tag{19}$$

In the boundary layer, R varies slowly, logarithmically (2). Neglecting this variation, we obtain from this $Q \sim 1/y$. According to (17), then also $\partial U_x/\partial y \sim 1/y$, which leads to the logarithmic profile of the mean velocity characteristic of a turbulent boundary layer: $U_x \sim \log y$.

Dimensionless constants have entered our equations, and they must be determined from experiment. The constants β and β_1, entering equations (2) and (3) for R_{ij} and S_{ijk}, were determined in (2) on the basis of measurements by Laufer (4). In doing so we used only measurements of the mean velocities U_i and the second moments R_{ij}, as the more reliable ones. It was found that $\beta \cong \beta_1 \cong 10$.

To determine the constants α and β_2 in equations (12) and (15) for Q and C_i, we shall use the results of the same measurements. The constant β_2 can be determined with the aid of the equalities (12) and (18) from data pertaining to the middle of the flow, in exactly the same way as in (2) the constant β_1 was determined on the basis of the energy equation (2a). The constant α then drops out. As a result, with satisfactory accuracy one obtains: $\beta_2 \cong 3$. The constant α, on the other hand, is determined from the same equation (12) from data pertaining to the turbulent boundary layer. This gives $\alpha \cong 3.3$. Thus, in essence, only two empirical constants enter our equations: $\alpha = 3.3$, $\beta = 10$. For a more accurate determination of them, and in general for a more complete verification of the equations themselves, special experiments are desirable.

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1. B. I. Davydov, DAN, **127**, 768 (1959).

2. B. I. Davydov, DAN, **127**, 980 (1959).

3. J. K. Batchelor, *The Theory of Homogeneous Turbulence*, 1955.

4. J. Laufer, NASA Rep., 1174 (1954).

Note: Figure translations are in progress. See original paper for figures.