ON THE TYPE OF GLUING OF A STRIP

1961

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Abstract

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MATHEMATICS

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ON THE TYPE OF GLUING OF A STRIP

(Presented by Academician M. A. Lavrent' ev, 1 II 1961)

1. In the present note criteria are indicated for the parabolic type of gluing of a strip, giving a strengthening of the results obtained in ([1,2]). Let $\bar{x} = g(x)$ be a continuously differentiable topological correspondence between the points of the boundary lines of the strip $S : 0 < y < 1, \ -\infty < x < \infty$ in the plane $z = x + iy$, admitting a quite definite conformal gluing ([3]). The function $g(x)$, as is customary, will be called the gluing function, and by S with the indicated correspondence on the boundary we shall understand a certain Riemann surface $S(g)$. Suppose that the function $w = T(z)$ effects a conformal sealing-up of $S(g)$, transforming the latter into a domain filling a disk $|w| < R \leq \infty$ one-sheetedly, in such a way that $T(x) = T[g(x) + i]$. Then $S(g)$ is called a surface of parabolic or hyperbolic type according as $R = \infty$ or $R < \infty$. Let us note that if $g_1(x)$ and $g_2(x)$ are two gluing functions of the strip satisfying, for all x, the conditions

$$\max\left(\frac{g_1'(x)}{g_2'(x)}, \ \frac{g_2'(y)}{g_1'(x)}\right) < K, \qquad |g_1(x) - g_2(x)| < K, \tag{1}$$

where K is a constant, then one can construct a q-quasiconformal mapping which reduces the problem of gluing S with gluing function $g_1(x)$ to the problem of gluing S with gluing function $g_2(x)$. In this case the type of $S(g_1)$ and $S(g_2)$ is one and the same.

2. Let $\varphi(x)$, $\varphi'(x) > 0$, be a continuous piecewise-linear function, defined for all $x \in (-\infty, \infty)$ and such that: a) the set M of abscissae of the break points of $\varphi(x)$ has the only limit point at ∞, and, for $x' \notin M$, $\varphi(x') = g(x')$; b) for all x the conditions (1) are fulfilled for $\varphi(x)$ and $g(x)$. The final choice of $\varphi(x)$ will be specified below. Denote $\{x_k\} = M \cap [0, \infty)$, $x_0 = 0$, $x_k < x_{k+1}$, $k = 0, 1, 2, ...$, and let γ_k be the rectilinear segments in S joining the points x_k and $\varphi(x_k) + i = \varphi_k + i$. Let $\{B_k\}$ be quadrilaterals, each of which is bounded by a pair of segments γ_k, γ_{k+1} and by segments of the boundary lines of S lying between them. Put $n_k = x_{k+1} - x_k$, $m_k = \varphi_{k+1} - \varphi_k$, and $S_0 = \bigcup_{k=0}^{\infty} B_k$. The modulus $\tilde{\mu}$ of the doubly connected domain $S_0(\varphi)$, conformally equivalent to S_0 with gluing function $\varphi(x)$, is obviously connected with the modulus of the part $S_0(g)$ of the surface $S(g)$, obtained from it by removing a certain disk, by the inequality

$$\frac{1}{K}\,\mu < \mu[S_0(g)] < K\tilde{\mu}.$$

This follows from the remark in § 1 and the choice of $\varphi(x)$. Let $\{B_k\} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, where
$\mathfrak{M}_1 = \mathfrak{M}[B_k,\ \max(m_k/n_k,\ n_k/m_k) \geq 1+\delta]$ and
$\mathfrak{M}_2 = \mathfrak{M}[B_k,\ \max(m_k/n_k,\ n_k/m_k) < 1+\delta]$, $\delta > 0$. For each $B_k \in \mathfrak{M}_1$ (for simplicity we omit the indices) with vertices a, b, c, and d (see Fig. 1, II) we indicate a quasiconformal gluing which transforms it into a ring. Extend the lateral sides of B_k until they meet at the point o. Assuming that the angle at the vertex o is small, we shall distinguish two cases: 1) $|\pi/2-\theta| < \theta_0$ and 2) $|\pi/2-\theta| \geq \theta_0 > 0$, where θ is the angle at the vertex c. In case 2), assuming for definiteness that $\overline{bc} > \overline{ad}$ and $\theta \leq \pi/2-\theta_0$, we fix on the side dc points p_1 and p_2 such that $\overline{bp_1} = \overline{bc}$ and $\overline{ob} = \overline{op_2}$.

One can construct a quasiconformal mapping $\tau = \mu_{k'}(z)$, taking the triangle p_1bc onto the triangle p_1bp_2, which leaves the side p_1b fixed, $\mu_{k'}(c) = p_2$, and has constant stretching at the points of the side bc. Since in the indicated triangles the angles at the vertex b satisfy the condition $\min(\theta', \theta'') > \theta_0$, the characteristic of this mapping is $p_1(z) < K_1$, where K_1 depends only on θ, and $K_1(\theta) \to \infty$ as $\theta \to 0$. Let now q_1 and q_2 be points on the side ab such that $\overline{ad} = \overline{dq_1}$ and $\overline{od} = \overline{oq_2}$, and let $\tau = \chi_{k'}(z)$ be an analogous quasiconformal mapping taking the triangle adq_1 onto the triangle q_2dq_1. Here the side dq_1 remains fixed, $\chi_{k'}(a) = q_2$, and $|\chi_{k'}(z)| = \text{const}$ at the points of the side ad. By the same considerations its characteristic $p_2(z) < K_2$, where $K_2(\theta) \to \infty$ as $\theta \to 0$. The function $\tau = f_{k'}(z)$, $f_{k'}(z) = \mu_{k'}(z)$ for $z \in \Delta_{k'} = (p_1bc)$, $f_{k'}(z) = \chi_{k'}(z)$ for $z \in \Delta_{k'} = (adq_1)$, and $f_{k'}(z) \equiv z$ for the remaining points $B_{k'}$, realizes a quasiconformal mapping of $B_{k'}$ onto the quadrilateral $\hat{B}_{k'} = (q_2dp_2b)$ with piecewise-continuous characteristic $p(z)$, $p(z) < \max(K_1, K_2)$. Let σ_1 and σ_2 be smooth Jordan arcs in $\hat{B}_{k'}$ with endpoints at the points q_2, d and p_2, b, respectively, dividing $\hat{B}_{k'}$ into the regions $\mathfrak{B}'_{k'}$, $\mathfrak{B}''_{k'}$, and $\hat{B}_{k'} \setminus (\mathfrak{B}'_{k'} \cup \mathfrak{B}''_{k'})$. Choosing σ_1 and σ_2 so that the angles which they form with the segments q_2d and p_2b are, for all k', no smaller than a certain positive constant, and moreover $\bar{\sigma}_1 = O(\overline{q_2d})$, $\bar{\sigma}_2 = O(\overline{p_2b})$, it is easy to construct in $\hat{B}_{k'}$ an additional q-quasiconformal deformation $\tau^* = t(\tau)$, which in $\hat{B}_{k'} \setminus (\mathfrak{B}'_{k'} \cup \mathfrak{B}''_{k'})$ coincides with the identity transformation, while inside $\mathfrak{B}'_{k'}$ and $\mathfrak{B}''_{k'}$ it takes, with constant stretching, the boundary segments q_2d and p_2b to arcs of circles of radii $\overline{od}$ and $\overline{op_2}$, respectively. By means of the superposition $w = h(\tau^*)$—a conformal compression with coefficient $1/\overline{ob}$, a logarithm, and an integral linear function —we pass to a vertical rectangle $G_{k'}$ in the plane w, $w = u + iv$, of unit height and base $(u_{k'}, u_{k'+1})$. The transformation $w^* = \exp(2\pi w)$ turns $G_{k'}$ into an annulus. The indicated gluing is preserved for $\theta \leq \pi/2 - \theta_0$, $\overline{bc} < \overline{ad}$, and also for $\theta \geq \pi/2 + \theta_0$, $\overline{bc} \leq \overline{ad}$. In all cases the length of the base of the rectangle $G_{k'}$, as an elementary calculation shows, satisfies the inequality

$$d_{k'} > K\,\frac{[\max(m_{k'}/n_{k'},\, n_{k'}/m_{k'}) - 1]\min(n_{k'}, m_{k'})}{\min(\gamma_{k'}^2, \gamma_{k'+1}^2)\ln\max(m_{k'}/n_{k'},\, n_{k'}/m_{k'})}, \quad \text{where } K \text{ is a constant.} \tag{2}$$

Consider now case 1). The preliminary quasiconformal transformation $f_{k'}(z)$ is defined here differently. Assuming that $\pi/2 - \theta_0 \le \theta \le \pi/2$, $\overline{ad} < \overline{dc}$ and $\overline{bc} > \overline{ad}$, mark on the side ab the point p_1, $\overline{p_1 b} = \overline{bc}$, and map quasiconformally, by means of the function $\tau = \tilde{\mu}_{k'}(z)$, the triangle p_1bc onto the triangle p_1p_2c (Fig. 1, III), where p_2 is a point lying on the continuation of the side ab, $\overline{op_2} = \overline{oc}$. We choose this mapping so that the

[Fig. 1]

Fig. 1

the side p_1c remained fixed, $\tilde{\mu}_{k'}(b) = p_2$, and $|\tilde{\mu}_{k'}(z)| = \text{const}$ at the points of the side bc. Since the angles of these triangles are uniformly bounded below by a positive constant, $\tilde{\mu}_{k'}(z)$ is a q-quasiconformal mapping. Next, in the triangle adq_1 we define a q-quasiconformal mapping $\tilde{\chi}_{k'}(z)$ carrying it onto the triangle aq_2q_1, where q_1 and q_2 are points lying respectively on dc and od, $\overline{ao} = \overline{oq_2}$ and $\overline{ad} = \overline{dq_1}$. In this case the side aq_1 remains fixed, $\tilde{\chi}_{k'}(d) = q_2$, and $|\tilde{\chi}_{k'}(z)| = \text{const}$ at the points of the side ad. We now put $f_{k'}(z) = \tilde{\mu}_{k'}(z)$ for $z \in \Delta_{k'} = (p_1bc)$, $f_{k'}(z) = \tilde{\chi}_{k'}(z)$ for $z \in \Delta'_{k'} = (adq_1)$, and $f_{k'}(z) \equiv z$ for the remaining points. The further transformations are the same as in case 2). For all possible forms here of the original quadrilateral $B_{k'}$, the estimate for the length of the base $d_{k'}$ is the same as in (2).

Let $G_{k''}$ be a rectangle, conformally equivalent to $B_{k''} \in \mathfrak{M}_2$, of unit height, with base $(u_{k''}, u^0_{k''+1})$ and with the identical correspondence between the points of the horizontal bases. We may assume that all $\{G_{k'}, G_{k''}\} = \{G_k\}$ fill, in a one-to-one manner, the rectangle R, $0 < u < \rho \le \infty$, $0 < v < 1$, and that between the points of the adjacent sides of each pair of rectangles G_k and G_{k+1}, $k = 0, 1, 2, ...$, a homeomorphic correspondence is established. Assigning to each point u_k the interval (u_k, u_k^0), $u_k^0 \in (u_k, u_{k+1})$, where (u_k, u_{k+1}) is the base of G_k, we define a quasiconformal gluing of $\{G_k\}$, identical outside the rectangles $\widehat{G}_k = \{u \in (u_k, u_k^0),\ v \in (0,1)\}$, and inside quasiconformal, preserving the abscissae of the points. Put

$$I_n = \bigcup_{k=1}^{n} (u_k, u_k^0).$$

Let $w = H(z)$ be the resulting homeomorphic mapping of S_0 onto R, quasiconformal with characteristic $p(w)$. If $R_n = \{u \in (0, u_{n+1}),\ v \in (0,1)\}$, then, by Grötzsch’ s principle,

$$\mu(R_n) > \sum_{k=1}^{n} \mu(G_k \setminus \widehat{G}_k)$$

and, consequently, (4),

$$\mu(R_n) > \sum_{(k')} \int_{u^0_{k'}}^{u_{k'+1}} \frac{du}{\displaystyle\int_{\Gamma_u} p\,\frac{du}{dn}\,ds} + \sum_{(k'')} \int_{x'_{k''}}^{x''_{k''}} \frac{dx}{\displaystyle\int_{\Gamma_x} \frac{dx}{dn'}\,ds'}. \tag{3}$$

Here Γ_u are vertical segments in $G_{k'} \subset R_n$, and dn and ds are the elements of the normal and of arclength on Γ_u. The section Γ_x coincides with the rectilinear segment in $B_{k''}$ joining the points x and $\varphi(x)+i$; dn' and ds' are the elements of the normal and of arclength on Γ_x, and $(x'_{k''}, x''_{k''})$ is the base of the quadrilateral formed by all segments $\Gamma_x \subset B_{k''} \subset H^{-1}(R_n)$ for which $\Gamma_x \cap H^{-1}(G_{k''}) = 0$. On such a segment (Fig. 1, I) we have

$$dx/dn' \le \sqrt{1 + [\varphi(x) - x]^2}\,\{1 + [\varphi'(x)]^{-1}\},$$

therefore

$$\sum_{(k'')} \int_{x'_{k''}}^{x''_{k''}} \frac{dx}{\displaystyle\int_{\Gamma_x} \frac{dx}{dn'}\,ds'} \ge \frac{1}{2} \sum_{(k'')} \int_{x'_{k''}}^{x''_{k''}} \frac{\min(dx, d\varphi)}{1 + [\varphi(x) - x]^2}. \tag{4}$$

Since on Γ_u, $du/dn = 1$, putting $\Gamma'_u = \Gamma_u \cap H(\Delta'_{k'} \cup \Delta_{k'} \cup \mathfrak{B}'_{k'} \cup \mathfrak{B}''_{k'})$, we have, on $\Gamma_u \subset G_{k'} \setminus \widehat{G}_{k'}$,

$$\int_{\Gamma_u} p(w)\frac{du}{dn}\,ds = \int_{\Gamma'_u} p(w)\frac{du}{dn}\,ds + \int_{\Gamma_u \setminus \Gamma'_u} \frac{du}{dn}\,ds < 1 + \int_{\Gamma'_u} p(w)\,ds. \tag{5}$$

Since $|dh/d\tau^*| < 1$, it follows that

$$\int_{\Gamma'_u} p(w)\,ds < \max(K_1, K_2) \int_{\Gamma'_u} ds < K \max(K_1, K_2) \max(m_{k'}, n_{k'}),$$

where K is a constant independent of the choice of the section Γ_u. Consequently, the right-hand side in (5) is uniformly bounded for a suitable choice of the function $\varphi(x)$. From (3), (4) there then follows the inequality

$$\mu(R_n) > K' \sum_{(k')} \int_{u^0_{k'}}^{u_{k'}+1} du + \frac{1}{2} \sum_{k''} \int_{x'_{k''}}^{x_{k''}{}''} \frac{\min(dx, d\varphi)}{1 + [\varphi(x) - x]^2}. \tag{6}$$

Let $\varphi(x)$ be chosen so that the oscillation of $1+[\varphi(x)-x]^2$ on each interval (x_k, x_{k+1}) does not exceed $K = \text{const}$, and, for all $x \in (-\infty, \infty)$,

$$\frac{1}{K}\{1+[\varphi(x)-x]^2\} < 1+[g(x)-x]^2 < K\{1+[\varphi(x)-x]^2\}. \tag{7}$$

Then from (6), (7), and (2), taking into account that

$$\min(\bar{\gamma}_k^2, \bar{\gamma}_{k+1}^2) = \min\{1+[\varphi(x_k)-x_k]^2,\ 1+[\varphi(x_{k+1})-x_{k+1}]^2\}$$

and that the stretching of the mapping $H(z)$ is constant at the points of the horizontal bases $B_{k'}$, we obtain

$$\mu(R_n) > K''\left\{\int_{E_n'} \frac{\max[g'(x),\, 1/g'(x)]\,\min(dx, dg)}{[1+\psi^2(x)]\,\ln\max[g'(x),\, 1/g'(x)]} + \int_{E_n''} \frac{\min(dx, dg)}{1+\psi^2(x)}\right\} + \varepsilon_n, \tag{8}$$

where $\psi(x) = g(x) - x$, $E_n' = E[x,\ \max(\varphi',\ 1/\varphi') \geq 1+\delta] \cap H^{-1}(\overline{R_n})$, $E_n'' = E[x,\ \max(\varphi',\ 1/\varphi') < 1+\delta] \cap H^{-1}(\overline{R_n})$, and ε_n depends on the choice of I_n, with $\lim \varepsilon_n = 0$ as $\operatorname{mes} I_n \to 0$. The sum of the integrals on the right-hand side of (8), up to a term uniformly bounded as $n \to \infty$ and depending only on the choice of $\varphi(x)$, coincides with the sum of integrals of the same kind taken over the sets $\mathcal{E}_n' = \mathcal{E}[x,\ \max(g',\ 1/g') \geq 1+\delta] \cap H^{-1}(\overline{R_n})$ and $\mathcal{E}_n'' = \mathcal{E}[x,\ \max(g',\ 1/g') < 1+\delta] \cap H^{-1}(\overline{R_n})$. Passing to the limit as $n \to \infty$ and observing that $\lim \mu(R_n) = \mu(R)$, we arrive at the following assertion.

Theorem. For the parabolic type of $S(g)$, it is sufficient that at least one of the integrals diverge

$$\int_{\mathcal{E}'} \frac{\max(dx, dg)}{[1+\psi^2(x)]\,\ln\nu(x)}, \qquad \int_{\mathcal{E}''} \frac{\min(dx, dg)}{1+\psi^2(x)},$$

where $\psi(x) = g(x) - x$, $\nu(x) = \max(g', 1/g')$, and $\mathcal{E}'$ and $\mathcal{E}''$ are the sets on which, respectively, $\nu(x) \geq 1+\delta$ and $\nu(x) < 1+\delta$, $\delta > 0$.

As a consequence of this theorem we obtain a result of L. I. Volkovyskii.

Corollary 1. For the parabolic type $S(g)$, it is sufficient that the integral

$$\int_0^\infty \frac{\min(dx, dg)}{1+\psi^2(x)}$$

diverge.

Corollary 2. Let $g(x)$ be such that $\psi(x) \leq O(1)$. Then, for the parabolic type $S(g)$, it is sufficient that at least one of the integrals diverge

$$\int_{\mathcal{E}'} \frac{\max(dx, dg)}{\ln \nu(x)}, \qquad \int_{\mathcal{E}''} \min(dx, dg). \tag{9}$$

Let us note that under the hypotheses of Corollary 2 there exist functions leading to the hyperbolic type $S(g)$. At the same time it is not known whether, in this case, the simultaneous convergence of the integrals (9) is a sufficient condition for the hyperbolic type $S(g)$.

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Received
25 I 1961

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Note: Figure translations are in progress. See original paper for figures.