MATHEMATICS

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Abstract

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ON THE CONVERGENCE OF A SEQUENCE OF CONTINUOUS FUNCTIONS TO INFINITY

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Let $f_n(x)$ be a sequence of continuous functions defined on the space $\mathcal{E}$ of real numbers. Let

$$F_1 = \{x : \lim_{n\to\infty} f_n(x) = +\infty\}; \tag{1}$$

$$F_2 = \{x : \lim_{n\to\infty} f_n(x) = -\infty\}. \tag{2}$$

Then, as is known (see (^{1}), p. 259),

$$F_1 = \bigcap_{m=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{n=j}^{\infty}\{x : f_n(x) \geq m\}; \qquad F_2 = \bigcap_{m=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{n=j}^{\infty}\{x : f_n(x) \leq -m\}. \tag{3}$$

Since the sets $\{x : f_n(x) \geq m\}$, $\{x : f_n(x) \leq -m\}$ are closed, it follows that

$$F_1 \in F_{\sigma\delta}, \qquad F_2 \in F_{\sigma\delta}. \tag{4}$$

It is obvious that

$$F_1 \cap F_2 = 0. \tag{5}$$

Let $F \in F_{\sigma\delta}$. Hahn (^{2}), and also Sierpiński (^{3}), proved that there exists a sequence of continuous functions $f_n(x)$ such that $F = \{x : \lim_{n\to\infty} f_n(x) = 0\}$ (see also (^{1}), pp. 261–262).

Put $\varphi_n(x) = [\sup(n^{-1}, |f_n(x)|)]^{-1}$. Then we have $F = \{x : \lim_{n\to\infty} \varphi_n(x) = +\infty\}$. Thus we see that the set of all those points at which a sequence of continuous functions $f_n(x)$ converges to $+\infty$ has type $F_{\sigma\delta}$. Conversely, every set of type $F_{\sigma\delta}$ is the set of convergence to $+\infty$ of some sequence of continuous functions.

The question arises: if conditions (4) and (5) are satisfied, does there exist a sequence of continuous functions $f_n(x)$ for which (1) and (2) hold? I. P. Kornfeld observed that the answer to this question is negative. Indeed, the sets

$$F_1 \subset E^{(1)} = \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} \{x : f_n(x) \geq 1\},$$

$$F_2 \subset E^{(2)} = \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} \{x : f_n(x) \leq -1\}.$$

It is obvious that $E^{(1)} \in F_\sigma$, $E^{(2)} \in F_\sigma$, and $E^{(1)} \cap E^{(2)} = 0$. Using the terminology of N. N. Luzin, we may conclude that the sets F_1 and F_2 must be separable by sets of type F_σ. I. P. Kornfeld also found some sufficient conditions for the pair of sets F_1, F_2, but they differ from the necessary ones.

In the present paper we prove that the indicated necessary conditions are sufficient, i.e. *for any two sets F_1 and F_2 of type $F_{\sigma\delta}$ and separable by F_σ, there exists a sequence of continuous functions $f_n(x)$ for which (1) and (2) hold.*

First we shall formulate several lemmas on sets of types F_σ and $F_{\sigma\delta}$; then we shall define the functions $f_n(x)$ and give a brief outline of the proof of the assertion stated above.

Lemma 1. Let $A \in F_\sigma$. Then there exist sequences of open sets P_n and closed bounded sets K_n such that

$$K_n \subset K_{n+1}, \quad K_n \subset P_n, \quad A = \bigcup_{n=1}^{\infty} K_n = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \overline{P}_n.$$

This lemma is first proved for the case when A is a set of type F_σ and of the first category; then when A is an open set. In the second case one obtains the result

$$A = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \overline{P}_n$$

(more precise than

$$A = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \overline{P}_n).$$

Since an arbitrary set $A \in F_\sigma$ is the sum of an open set and a set F_σ of the first category, from the validity of the lemma for the special cases one can prove its validity also in the general case.

Lemma 2. Let $E^{(1)} \in F_\sigma$, $E^{(2)} \in F_\sigma$, and $E^{(1)} \cap E^{(2)} = 0$. Then there exist open sets $L_j^{(i)}$ $(i = 1, 2;\ j = 1, 2, \ldots)$ and closed bounded sets $K_j^{(i)}$, satisfying the conditions

$$\overline{L}_j^{(1)} \cap \overline{L}_j^{(2)} = 0, \quad K_j^{(i)} \subset K_{j+1}^{(i)}, \quad K_j^{(i)} \subset L_j^{(i)}, \quad E^{(i)} = \bigcup_{j=1}^{\infty} K_j^{(i)} = \lim_{j\to\infty} L_j^{(i)} = \lim_{j\to\infty} \overline{L}_j^{(i)}.$$

Lemma 3. Let

$$E_1 = \bigcup_{i=1}^{\infty} F_i,$$

where $F_i \subset F_{i+1}$ are closed bounded sets, and let G_i be open sets such that $F_i \subset G_i$,

$$E_1 = \lim_{i\to\infty} G_i.$$

Then, if $E_2 \in F_\sigma$ and $E_2 \subset E_1$, there exists a sequence of open sets B_i and a sequence of closed bounded sets H_i such that

$$H_i \subset H_{i+1}, \quad H_i \subset B_i, \quad \overline{B}_i \subset G_i, \quad E_2 = \bigcup_{i=1}^{\infty} H_i = \lim_{i\to\infty} B_i = \lim_{i\to\infty} \overline{B}_i.$$

For the proof of Lemmas 2 and 3, Lemma 1 is used.

Lemma 4. Let $F_i \in F_{\sigma\delta}$ $(i = 1, 2)$, $E^{(i)} \in F_\sigma$, $F_i \subset E^{(i)}$, and $E^{(1)} \cap E^{(2)} = 0$. Then there exist open sets $L_{j,k}^{(i)}$ $(j = 1, 2, \ldots;\ k = 1, 2, \ldots)$ such that

$$\overline{L}_{1,k}^{(1)} \cap \overline{L}_{1,k}^{(2)} = 0, \quad L_{j,k}^{(i)} \supset \overline{L}_{j+1,k}^{(i)},$$

$$F_i = \lim_{j\to\infty} \lim_{k\to\infty} \overline{L}_{j,k}^{(i)}. \tag{6}$$

For the proof of the lemma, first Lemma 2 is used, and then Lemma 3.

Theorem. Let for a pair of sets $F_i \in F_{\sigma\delta}$ $(i = 1, 2)$ there exist a pair of sets $E^{(i)} \in F_\sigma$, satisfying the conditions $F_i \subset E^{(i)}$, $E^{(1)} \cap E^{(2)} = 0$. Then there exists a sequence of continuous functions $f_n(x)$ such that equalities (1) and (2) are fulfilled.

Let the sets $L_{j,k}^{(i)}$ $(i = 1, 2)$ have the properties indicated in Lemma 4. Then the sets

$$\overline{L}^{(1)}_{n,n}, \quad \overline{L}^{(2)}_{n,n}, \quad \overline{L}^{(1)}_{j,n} \setminus L^{(1)}_{j,n}, \quad \overline{L}^{(2)}_{j,n} \setminus L^{(2)}_{j,n} \quad (j = 1, 2, \dots, n-1)$$

are closed and have no pairwise common points. Put

$$f_n(x) = \begin{cases} (-1)^{i+1} n, & \text{for } x \in \overline{L}^{(i)}_{n,n}, \\ (-1)^{i+1} j, & \text{for } x \in \overline{L}^{(i)}_{j,n} \setminus L^{(i)}_{j,n}, \end{cases}$$

where $i = 1, 2;\ j = 1, 2, \dots, n-1$. In each component interval of the open set

$$\mathcal{E} \setminus (\overline{L}^{(1)}_{1,n} \cup \overline{L}^{(2)}_{1,n}),$$

whose boundary points belong to two different sets $\overline{L}^{(1)}_{1,n}$ and $\overline{L}^{(2)}_{1,n}$, the function $f_n(x)$ is defined so that

so that in the closure of this component interval it is linear. In a component interval whose boundary points belong to the same set $\overline{L}^{(i)}_{1,n}$, the function $f_n(x)$ is defined by the equality

$$f_n(x) = (-1)^{i+1} + (-1)^i \min[1/2, \rho(x, \overline{L}^{(i)}_{1,n})].$$

Since at the boundary points of this component interval the function $f_n(x)$ has the value $(-1)^{i+1}$, $f_n(x)$ is continuous in the closure of the interval. Moreover, on this component interval the function $f_n(x)$ satisfies the Lipschitz condition with constant 1. It remains now to define $f_n(x)$ at the points of the sets $L^{(i)}_{j-1,n} \setminus \overline{L}^{(i)}_{j,n}$ $(1 < j \leq n)$. These sets are open, and at the boundary points of their component intervals the function $f_n(x)$ has already been defined. If the boundary points of an interval belong to two different sets $\overline{L}^{(i)}_{j-1,n}$, $\overline{L}^{(i)}_{j,n}$, or both belong to the set $\overline{L}^{(i)}_{j-1,n}$, then $f_n(x)$ is defined so that it is linear in the closure of the interval. If both boundary points belong to $\overline{L}^{(i)}_{j,n}$, then let, in this component interval,

$$f_n(x) = (-1)^{i+1} j + (-1)^i \min[1/2, \rho(x, \overline{L}^{(i)}_{j,n})].$$

Thus, in the closure of an arbitrary component interval the function $f_n(x)$ is either linear or satisfies the Lipschitz condition with constant 1.

It is possible to verify that all the functions $f_n(x)$ are continuous and that

$$E^{(1)}_{m,n} = \{x : f_n(x) \geq m\} = \begin{cases} \overline{L}^{(1)}_{m,n}, & \text{for } 1 \leq m \leq n, \\ 0, & \text{for } m > n; \end{cases}$$

$$E^{(2)}_{m,n} = \{x : f_n(x) \leq -m\} = \begin{cases} \overline{L}^{(2)}_{m,n}, & \text{for } 1 \leq m \leq n, \\ 0, & \text{for } m > n. \end{cases}$$

Hence it follows that

$$\lim_{n\to\infty} E_{m,n}^{(i)} = \lim_{n\to\infty} \overline{L}_{m,n}^{(i)}.$$

On the basis of (3) and of the preceding,

$$\{x : \lim_{n\to\infty} f_n(x) = +\infty\} = \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} E_{m,n}^{(1)} = \lim_{m\to\infty} \lim_{n\to\infty} \overline{L}_{m,n}^{(1)},$$

$$\{x : \lim_{n\to\infty} f_n(x) = -\infty\} = \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} E_{m,n}^{(2)} = \lim_{m\to\infty} \lim_{n\to\infty} \overline{L}_{m,n}^{(2)}.$$

Taking (6) into account, we see that equalities (1) and (2) are valid for the constructed sequence of functions $f_n(x)$, as was required to prove.

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Note: Figure translations are in progress. See original paper for figures.

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