Mathematics

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1961

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Abstract

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ON THE REDUCIBILITY OF A CONTINUOUS FUNCTION OF A REAL VARIABLE AND THE STRUCTURE OF THE FIXED POINTS OF THE CORRESPONDING ITERATION PROCESS

(Presented by Academician N. N. Bogolyubov, IV 1 1961)

Let $f(x)$ be a real function defined on the entire real axis R. We shall say that a set $M \subseteq R$ reduces the function $f(x)$ if $f(M) \subseteq M$ and $f(R \setminus M) \cap M = 0$.

The reducibility of a function is, naturally, closely connected with the structure of the iteration process generated by this function, i.e., with the domains of attraction of fixed points and their boundaries. And the latter, in the case of a function of one real variable, as will be seen below, are completely determined by the fixed points and by the points that pass into them, i.e., by the zeros of the functions

$$f_{k+m}(x) - f_m(x), \qquad k = 1, 2, \dots; \quad m = 0, 1, 2, \dots,$$

where $f_i(x) = f_{i-1}(f(x));\ i = k + m,\ m.$

As is known, a point α is called a fixed point of order k if $f_k(\alpha) = \alpha$ and $f_j(\alpha) \neq \alpha$ for $1 \leq j < k$. The fixed points of order k of the iteration process defined by a continuous function form a closed set which, like any closed set, may consist of a perfect part and of at most a countable number of points. Points of the second kind of a perfect set usually do not play an essential role in the structure of the iteration process, and in what follows, for simplicity, we shall assume that the set of fixed points has no perfect part. Then the set of fixed points of all orders will be at most countable. It is further known that there exist attracting, repelling, and so-called indifferent fixed points. To the latter belong points that are limit points for isolated fixed points, and fixed points that are points of the second kind of perfect sets,* which, as we have agreed, are absent in our case. Finally, let us note that the character of a fixed point may be different on the two sides of it; for example, on one side it may appear as an attracting point, and on the other as a repelling one.

The following three theorems are devoted to the structure of the set of fixed points, by which, generally speaking, is meant the dependence between the existence of fixed points of different orders, and the dependence between the existence of attracting and repelling fixed points.

In ([2]) it was shown that if an iteration process defined by a continuous function has a fixed point of order $k > 2$, then it also has a fixed point of the second order. Hence it follows at once:

Theorem 1. *If the iteration process defined by a continuous function $f(x)$ has no fixed points of order 2^m, then it can have only fixed points of order 2^i, $i = 0, 1, \dots, m-1$. If, however, the ite-*

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* In ([1]) every fixed point α of order k for which

$$\left|\frac{df_k(x)}{dx}\right|_{x=\alpha} = 1$$

is classified as indifferent.

rational process has a fixed point of order different from 2^j, $j = 0, 1, 2, \dots$, then there exist fixed points of arbitrarily high order.

Indeed, if there are no fixed points of order 2^m, then there are also no fixed points of order $s \cdot 2^{m-1}$, $s = 3, 4, \dots$, and the zeros of the functions $f_{s \cdot 2^{m-1}}(x) - x$, $s = 2, 3, \dots$, coincide with the zeros of the function $f_{2^{m-1}}(x) - x$. If the iteration process had a fixed point of order $l \neq 2^i$, $i = 0, 1, \dots, m-1$, then the function $f_l(x) - x$, and hence also the function $f_{l \cdot 2^{m-1}}(x) - x$, would have zeros not coinciding with the zeros of $f_{2^{m-1}}(x) - x$, which is impossible. If, however, the iteration process has a fixed point of order different from a power of two, for example of order 3, then it also has fixed points of order 2^j, $j = 0, 1, 2, \dots$, i.e. points of arbitrarily high order.

Theorem 2. *If $f(x)$ is continuous on R, then between any two attracting fixed points there always exists a repelling fixed point.*

If $f(x)$ is differentiable on R, then between any two repelling fixed points there exists at least either an attracting fixed point, or a countable number of repelling fixed points, among which there are also points of arbitrarily high order.

This theorem follows directly from the continuity of the function and from the fact that, if a fixed point of k-th order is attracting, then to the left of it $f_k(x) > x$ and to the right $f_k(x) < x$; if the fixed point is repelling, then, by differentiability, $f_{2k}(x) < x$ to the left and $f_{2k}(x) > x$ to the right of the fixed point.

One may assert the existence of fixed points having the required character only on one side.

The differentiability requirement is essential. Thus, the iteration process defined by the function

$$f(x) = \begin{cases} ax - x^2, & x \le a-1, \\ ax - (a-1)^2, & x \ge a-1, \end{cases} \qquad a > 2,$$

has only two fixed points, and both are repelling.

It remains to clarify the structure of the repelling fixed points.

Theorem 3. *The closure of the set of repelling fixed points may be an arbitrary closed set.*

In other words, the closure of the set of repelling fixed points may contain some perfect set and no more than a countable number of further points. Indeed, first of all, the iteration process may have isolated repelling fixed points, which may also accumulate. An example is provided by the iteration process defined by the function $f(x) = x + x^2 \sin \frac{1}{x}$. The indifferent fixed point $x = 0$ is a limit point for repelling (and attracting) isolated fixed points.

Next, the closure of the set of repelling fixed points may contain a perfect nowhere dense set. Apparently, when the iteration process has fixed points of arbitrarily high order, the latter form perfect, usually nowhere dense, sets. This, perhaps, is not difficult to establish, but for the proof of the theorem it is enough for us to give at least one example.

Let M be a closed interval, $f(M) \supset M$, $f(R \setminus M) \cap M = 0$, and let the set M^1 be such that $f(M^1) = M$, $f(M \setminus M^1) \cap M = 0$, consists of closed intervals, no fewer than two, and on each of them $f(x)$ is monotone. If there does not exist a set $\widetilde{M} \subset M$ such that $f(\widetilde{M}) \subseteq \widetilde{M}$ and $\operatorname{mes} \widetilde{M} > 0$, then M contains a perfect nowhere dense set M^0 such-

that $f(M^0) = M^0$, and which is the closure of the set of repelling fixed points.

The formulated conditions are satisfied, for example, by the functions $ax - bx^2$ for $M = [0, a/b]$ when $a > 4,\ b > 0$.

Take sets M^i such that $f_i(M^i) = M$ and

$$f_i\left(M \setminus \bigcup_{j=1}^{i} M^j\right) \cap M = 0,$$

$i = 2, 3, \dots.$ Since M^1 is a perfect set and contains at least two closed intervals, the sets M^i and

$$M^0 = \lim_{i \to \infty} M^i$$

are also perfect sets. We shall show that M^0 is the desired set. Clearly, $f(M^0) = M^0$ and M^0 is nowhere dense. Further, each closed interval $M^i_p \subset M^i,\ i, p = 1, 2, \dots$, contains at least one fixed point (repelling, since there are certainly no attracting ones on M, and belonging to M^0), because $f_i(M^i_p) = M$.

Finally, we shall show that the closure of the repelling fixed points can also be a closed interval. Let again M be a closed interval, $f(M) = M$, and suppose there is no set $\widetilde{M}$ such that $f(\widetilde{M}) \subseteq \widetilde{M}$ and $\operatorname{mes} \widetilde{M} > 0$. Then the fixed points fill M everywhere densely. Indeed, if M' is any closed interval belonging to M, then

$$\bigcup_{i=1}^{\infty} f_i(M') = M,$$

and hence there exists such a q that

$$\bigcup_{i=1}^{q} f_i(M') = M.$$

The set M, obviously, has at least one repelling fixed point, which, suppose, belongs to the set $f_r(M')$, $r \leq q$. There will always be a closed set $M'' \subseteq f_r(M')$, containing the indicated fixed point and such that $f(M'') \supset M''$. Similarly to what was said earlier, there exists such a p that

$$\bigcup_{j=1}^{p} f_j(M'') = M,$$

and, consequently, $f_p(M'') = M$. Thus, $f_{rp}(M') = M$, i.e. M' contains a repelling fixed point of order not exceeding rp.

A similar picture is obtained for $f(x) = 4x - bx^2,\ b > 0$, on $\left[0, \frac{4}{b}\right]$. Moreover, this case occurs for every function $f(x)$ such that $f(\varphi(x)) = \varphi(\theta(x))$, where $\varphi(x)$ is a continuous periodic function; $\theta(x)$ is a continuous function and such that for any closed interval $\widehat{M}$ there is an m such that the closed interval $\theta_m(\widehat{M})$ ($\theta_m(x)$ is the m-th iteration of $\theta(x)$) will contain at least one full period of the function $\varphi(x)$. For example, as $\theta(x)$ one may take a linear function $cx + d$ with $|c| > 1$. Setting $\theta(x) = nx,\ \varphi(x) = \cos x$, we obtain the Chebyshev polynomials $\cos(n \arccos x)$ $(M = [-1, 1])$. For the function

$$\frac{x^2 - 1}{2} = \operatorname{ctg}(2 \operatorname{arcctg} x)$$

the fixed points lie everywhere densely on the entire real axis.

We now formulate the following theorem on the reducibility of a continuous function of a real variable.

Theorem 4. *Let $f(x)$ be defined and continuous on R. Then R is decomposed into sets $M_i,\ i = 0, 1, 2, \ldots$, such that $f(M_i) \subseteq M_i$. The set M_0 is the closure of the set of repelling fixed points of the iteration process defined by $f(x)$, and of points going into them; each of the sets $M_1, M_2, \ldots$ is a domain of attraction to attracting fixed points of the k-th order that pass into one another, its boundary being fixed points of order not exceeding $2k$ and points going into them.*

We note that the sets $M_1, M_2, \ldots$ do not intersect. Each of them is open and has no points in common with M_0 if it contains two-sided attracting points, and is represented as the sum of an open set and some further set of points belonging to M_0, if it contains

one-sided attracting points. All the sets $M_0, M_1, M_2, \ldots$ reduce the function $f(x)$.

The question of the distribution of the points R among the sets M_i is sufficiently clear, and we shall not dwell on the proof of the theorem. The fact that the boundary of the domain of attraction of points of order k belongs to repelling fixed points of order not exceeding $2k$ was proved in ([2]).

The reducibility of a function of a real variable can be used in solving various equations which contain, as an argument of the unknown function, not only x, but also a certain known function $f(x)$, for example, in solving functional equations $\Phi(x, \varphi(x), \varphi(f(x))) = 0$, where $\varphi(x)$ is the unknown function.

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Received
28 III 1961

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Note: Figure translations are in progress. See original paper for figures.