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Abstract

Full Text

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ON INTEGRAL EQUATIONS CONTAINING INTEGRALS EXTENDED OVER SURFACES WITH EDGES

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1. Let $x(x_1, x_2, x_3)$, $y(y_1, y_2, y_3)$, $z(z_1, z_2, z_3)$, ... be points of three-dimensional Euclidean space; let l be a straight line passing through z, and let δ be a positive number. Denote by $H(l, z, \delta)$ the circular cylinder of height 2δ, whose axis is l, whose center of symmetry is the point z, and whose base radius is equal to δ.

Let S be a bounded open smooth manifold of the space under consideration. Denote its boundary by s, the tangent plane at the point z of the manifold S by $\tau(z)$, the positive normal at the point z by $n(z)$, and the part of S contained inside $H(n(z), z, \delta)$ by $S(z; \delta)$. Introduce a new coordinate system (Z_1, Z_2, Z_3) with origin at the point z. Direct the axis Z_3 along the normal $n(z)$, and choose Z_1 and Z_2 arbitrarily in the plane $\tau(z)$. We shall assume that s is a smooth curve and that there exists a positive number δ such that, for every point $z \in S + s$ (in what follows by S we shall mean the set $S + s$), the part of the manifold S, $S(z; \delta)$, admits a representation of the form $\eta_3 = g(\eta_1, \eta_2)$, where (η_1, η_2, η_3) are the coordinates of the point $y \in S(z; \delta)$ in the system (Z_1, Z_2, Z_3); $g(\eta_1, \eta_2)$ is a single-valued function on $\tau_s(z; \delta)$, while $\tau_s(z; \delta)$ is the orthogonal projection of $S(z; \delta)$ onto the plane $\tau(z)$. Suppose also that the first partial derivatives of the function $g(\eta_1, \eta_2)$ exist and belong to the class $H(\alpha)$ $(\alpha > 0)$ on $\tau_s(z; \delta)$.

Consider a function $k(x, y)$, defined on $S \times S$ and possessing the following property: if $x, y \in S(z; \delta)$, $x \neq y$, then $k(x, y)$ admits a representation of the form

$$k(x, y) = r^{-2}(x', y')\, f(x, \vartheta) + k_1(x, y), \tag{1}$$

where x' and y' are the orthogonal projections of the points x and y onto the plane $\tau(z)$; $r(x', y')$ is the distance between the points x' and y'; ϑ is the angle between the vector $\overrightarrow{x'y'}$ and the axis Z_1, measured from Z_1 counterclockwise; $f(x, \vartheta)$ satisfies the condition $H(\alpha)$ $(\alpha > 0)$ in both variables and, for every interior point x of the manifold S,

$$\int_0^{2\pi} f(x; \vartheta)\, d\vartheta = 0; \qquad k(x, y) = O\{r^{\alpha-2}(x', y')\}$$

uniformly with respect to x and y, and satisfies certain smoothness conditions –belongs, for example, to the class $N^{(\alpha,\lambda)}$ (see ([1])), where $\lambda > 0$.

A function $k(x, y)$ satisfying the requirements indicated above will be called a **singular kernel**. It is easy to construct an example of a singular kernel. Suppose a three-dimensional symmetric matrix $\|A_{ij}(x, y)\|$ is given on $S \times S$, where the functions $A_{ij}(x, y)$ $(i, j = 1, 2, 3)$ belong to the class $H(\lambda)$ on $S \times S$, and the quadratic form $\sum A_{ij}(x, y)t_i t_j$ is positive definite for every point $(x, y) \in S \times S$. It is easy to show ([2]) that the function

$$k(x, y) = \sum_{i=1}^{3} c_i(x, y)(x_i - y_i)\sigma^{-3}(x, y) + k_1(x, y), \tag{2}$$

$$\sigma^2 = \sum_{i,j=1}^{3} A_{ij}(x, y)(x_i - y_i)(x_j - y_j),$$

where $c_i(x, y)$ $(i = 1, 2, 3)$ are prescribed functions on $S \times S$, satisfying condition $H(\lambda)$; a singular kernel.

2. In this note we study integral equations of the form

$$A(\varphi) \equiv a(x)\varphi(x) + \int_S k(x, y)\varphi(y)\, dS_y = f(x), \tag{3}$$

where $a(x)$ and $f(x)$ are prescribed functions on S; $k(x, y)$ is a singular kernel; dS_y is the element of area of the manifold S at the point y; the integral is defined in the sense of the principal value (2); $\varphi(y)$ is the unknown function, which is sought in the class $L_p(S; \gamma)$ $(p > 1)$, and $\gamma(y)$ is a certain weight function.

Equation (3) in the space $L_p(S)$, when S is a closed surface, was studied by S. G. Mikhlin ([3,4]) (see also the works ([5–11])). The presence of the edge s of the manifold S (as, in the one-dimensional case, the presence of endpoints ([12,13])) complicates the study of equation (3) and requires other methods of investigation. We note that the method by means of which B. V. Khvedelidze ([13]) studies integral equations with Cauchy kernel for open contours has much in common with what will be described below.

3. We first give several auxiliary propositions. Denote by $\rho(x)$ the distance from the point x to the curve s, and consider the operators

$$K\varphi(x) = \int_S k(x, y)\varphi(y)\, dS_y, \qquad K_\varphi^\gamma(x) = \rho^{-\gamma}(x) \int_S k(x, y)\rho^\gamma(y)\varphi(y)\, dS_y,$$

where γ is an arbitrary real number, and the integrals at the points $y = x$ are defined in the sense of the principal value. For these operators the following theorems hold (cf. ([2,14])):

Theorem 1. If for every $\varphi(y) \in L_p(S)$ $(p > 1)$ there exists $K\varphi(x)$ for almost all x of S, $K\varphi$ is a bounded operator mapping $L_p(S)$ into itself, and $k(x, y) = O(r^{-2}(x, y))$, then $K_\varphi(x)$ exists for almost all x of S for every function $\varphi(y)$ from $L_p(S; \rho^\beta)$, where $-1 < \beta < p-1$, and is a bounded operator mapping $L_p(S; \rho^\beta)$ into itself.

Theorem 2. If $0 < \gamma < 1$, then, under the hypotheses of the preceding theorem, $K_\varphi^\gamma(x)$ exists for almost all x of S for every function $\varphi(y)$ from $L_p(S; \rho^{\gamma(p-1)})$ and is a bounded operator mapping $L_p(S; \rho^{\gamma(p-1)})$ into itself. If, however, $-1 < \gamma < 0$, then under the same hypotheses $K_\varphi^\gamma(x)$ exists for almost all x of S for every function $\varphi(y)$ from $L_p(S; \rho^\gamma)$ and is a bounded operator mapping $L_p(S; \rho^\gamma)$ into itself.

If $k(x, y)$ is a singular kernel, then the existence of $K\varphi(x)$ for almost all $x \in S$ and its boundedness in the space $L_p(S)$ follow easily from the works ([4,5,15]). Consequently:

Theorem 3. If $k(x, y)$ is defined by formula (1) (in particular, by formula (2)), then the assertions of Theorems 1 and 2 are valid.

Theorem 4. If $m(x, y)$ and $k(x, y)$, as well as their composition ([16]), are singular kernels, then for every $\varphi \in L_p(S; \rho^\beta)$ $(-1 < \beta < p-1)$ and almost all x of S the interchange formula is valid

$$\int_S m(x, y)\, dS_y \int_S k(y, z)\varphi(z)\, dS_z =$$

$$= c(x)\varphi(x) + \int_S \varphi(z)\, dS_z \int_S m(x, y)k(y, z)\, dS_z,$$

where $c(x)$ is a certain function independent of $\varphi(x)$.

4. Let us consider that special case of equation (3) in which $a(x) = 1$, $k(x, y)$ is defined by formula (2), $f \in L_p(S; \rho^{\alpha(p-1)})$ $(p > 1,\ 0 < \alpha < 1)$, and $\varphi(x)$ is the unknown function of the same class. Under these restrictions, instead of the operator A we shall write K^0.

First let us solve the following auxiliary problem: to determine a singular kernel $m(x, y)$ in such a way that

$$\omega(x, y) \equiv \rho^\alpha(x)k(x, y) + m(x, y)\rho^\alpha(y) + \int_S m(x, z)\rho^\alpha(z)k(z, y)\, dS_z = O(r^{\beta-2}(x, y)),$$

where β is some positive number.

This problem reduces to the solution of a singular integral equation with Hilbert kernel. The solution of this equation (which is written in explicit form) gives the characteristic function of the required singular kernel.

Let us now consider the operator

$$M^0(\varphi) = \varphi(x) + \rho^{-\alpha}(x)\int_S m(x,y)\rho^{\alpha}(y)\varphi(y)\,dS_y,$$

where $m(x,y)$ is the kernel constructed in the manner indicated above. Form the composition M^0K^0. From Theorems 1-3 of Section 3 it follows that M^0K^0 is a linear bounded operator mapping $L_p(S,\rho^{\alpha(p-1)})$ into itself. Further,

$$M^0K^0(\varphi) = [1+c(x)]\varphi(x) + \Omega_\varphi(x),$$

$$\Omega_\varphi(x) = \rho^{-\alpha}(x)\int_S \omega(x,y)\varphi(y)\,dS_y.$$

It is easy to see that $c(x) \neq -1$, and the operator $D\varphi(x) = [1+c(x)]\varphi(x)$ carries out a homeomorphic mapping of the space $L_p(S,\rho^{\alpha(p-1)})$ onto itself, while $\Omega_\varphi(x)$ is a completely continuous operator mapping $L_p(S,\rho^{\alpha(p-1)})$ into itself. Consequently, from a theorem of S. M. Nikol' skii ([17]) it follows that M^0K^0 is a Fredholm operator.

In an analogous manner one can construct a linear bounded operator N^0 of the form M^0, possessing the property that K^0N^0 is a Fredholm operator.

From what has been said above and from a theorem of F. V. Atkinson ([18]) and I. Ts. Gokhberg ([19]) it follows that K^0 is a Noether operator in the space $L_p(S,\rho^{\alpha(p-1)})$, i.e., for the equation $K^0\varphi = f$ the Noether theorems are valid.

5. We now indicate some generalizations.

1) Of the coefficients A_{ij} and c_i one may require only continuity.

2) One may consider the case of discontinuous coefficients. If, for example, A_{ij} and c_i undergo a discontinuity of the first kind along a smooth line l lying on S, then the reasoning of Section 4 will not change if by $\rho(x)$ we understand the distance from x to the set $l+s$.

3) The curves s and l may be piecewise smooth.

6. Let us now consider the general singular integral equation (3). The scheme indicated above for regularizing the equation $K^0(\varphi) = f$ can also be applied in a more general case, but the regularization problem is simplified if this scheme is modified somewhat and one uses the method for constructing a regularizer due to S. G. Mikhlin. For this purpose we somewhat extend the domain of definition of the characteristic $f(x;\vartheta)$. Namely, if $x \in S$ and $f(x;\vartheta)$ is not defined, then for $f(x,\vartheta)$ we take the value of the limit $\lim f(y;\vartheta)$ when $y \to x$ and $y \in S$. Now one may construct the symbol $\Phi(x,\vartheta)$ of the operator A for every point $x \in S$. Suppose that this symbol is everywhere different from

zero, and construct an operator B, whose symbol is equal to $\Phi^{-1}(x;\vartheta)$. From the theorem of S. G. Mikhlin ([4,10]) and from the assertions of item 3 it can be shown that the operator $\rho^{-\alpha}(x)B(\rho^{\alpha}\varphi)$ gives a regularization of equation (3) in the space $L_p(S,\rho^{\alpha(p-1)})$, and, consequently, Noether' s theorems are valid for this equation.

In conclusion we note that the arguments carried out above remain valid also when systems of singular equations are considered.

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Note: Figure translations are in progress. See original paper for figures.