# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# SOME THEOREMS ON DIFFUSION PROCESSES WITH A SMALL DIFFUSION COEFFICIENT AND THEIR APPLICATIONS TO SECOND-ORDER PARABOLIC EQUATIONS WITH A SMALL PARAMETER

*(Presented by Academician A. N. Kolmogorov on 20 II 1961)*

Let $x(t, \varepsilon, \omega)$ be a random process*, satisfying the equation

$$x(t, \varepsilon, \omega) = x_0 + \int_0^t a(u, x(u, \varepsilon, \omega))\, du + \varepsilon \int_0^t b(u, x(u, \varepsilon, \omega))\, d\xi(u, \omega), \quad (1)$$

where $u, t \in [0, T]$, $T < \infty$; $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 > 0$; $\xi(u, \omega)$ is a Wiener process. (For the theory of such equations see, for example, ([1]), Ch. VI, § 3.)

Below we shall always assume that the functions $a(u, x)$ and $b(u, x)$ satisfy the inequality

$$|a(u, x) - a(u, y)| + |b(u, x) - b(u, y)| \leq K|x - y|, \quad (2)$$

where $K < \infty$ uniformly in $u, x, y$, $u \in [0, T]$, $x, y \in R$. We shall assume that the Lipschitz condition (see (2)) is also satisfied by the first $m$ partial derivatives with respect to $x$ of the function $a(u, x)$ and by the first $(m - 1)$ partial derivatives with respect to $x$ of the function $b(u, x)$.

Under these assumptions, in the author' s paper ([2]) it was proved that

$$x(t, \varepsilon, \omega) = x(t) + \varepsilon x_1(t, \omega) + \cdots + \varepsilon^m x_m(t, \omega) + \varepsilon^{m+1} x_{m+1}(t, \varepsilon, \omega), \quad (3)$$

where $x(t)$ is the solution of the equation

$$x(t) = x_0 + \int_0^t a(u, x(u))\, du, \quad (4)$$

obtained from (1) for $\varepsilon = 0$. In ([2]), for the process

$$\tilde{x}_m(t,\omega) = (x_1(t,\omega), x_2(t,\omega), \dots, x_m(t,\omega))$$

stochastic integral equations of type (1) were found, and it was proved that the random process $x_{m+1}(t,\varepsilon,\omega)$ is uniformly bounded in the mean square, i.e.

$$\mathbf{M}\left\{\sup_{t\in[0,T]} |x_{m+1}(t,\varepsilon,\omega)|^2\right\} \le K < \infty$$

uniformly in $\varepsilon$.

From the theorems of paper ([2]) one can obtain limit theorems (asymptotics in $\varepsilon$) for

$$P^\varepsilon(t,x_0,x) = \mathbf{P}\{x(t,\varepsilon,\omega) < x \mid x(0,\varepsilon,\omega) = x_0\}$$

as $\varepsilon \to 0$. However, these results do not make it possible to obtain "local" limit theorems, i.e. theorems on the behavior of the densities

$$p^\varepsilon(t,x_0,x) = \frac{\partial}{\partial x} P^\varepsilon(t,x_0,x)$$

as $\varepsilon \to 0$.

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* All constructions are carried out in the probability space $(\Omega, \mathfrak{M}, P)$, where $\Omega$ is the set of elementary events $\omega$; $\mathfrak{M}$ is a $\sigma$-algebra of subsets of $\Omega$, on which a probability measure $P$ is defined. A random variable $x(\omega)$ is a $P$-measurable function with values in $R = (-\infty,\infty)$, and

$$\mathbf{M}x(\omega) = \int_\Omega x(\omega)\,P(d\omega).$$

Below we give several theorems on the asymptotics of $p^\varepsilon(t,x_0,x)$ and a number of auxiliary assertions.

Let us agree that, when considering the $n$-dimensional process $x(t,\varepsilon,\omega) = (x^1(t,\varepsilon,\omega), x^2(t,\varepsilon,\omega), \dots, x^n(t,\varepsilon,\omega))$, it is convenient to assume that $x = (x^1, x^2, \dots, x^n)$, $x \in R^n$; $a(u,x) = (a^1(u,x), a^2(u,x), \dots, a^n(u,x))$ is an $n$-dimensional vector; $b(u,x) = \{b^i_j(u,x)\}^n_{i,j=1}$ is a matrix; $\xi(u,\omega) = (\xi^1(u,\omega), \xi^2(u,\omega), \dots, \xi^n(u,\omega))$ is an $n$-dimensional Wiener process, and

$$|x| = \left(\sum_{i=1}^n |x^i|^2\right)^{1/2}, \qquad |b(u,x)| = \left(\sum_{i,j=1}^n |b^i_j|^2\right)^{1/2}.$$

In this notation everything said above remains valid in the $n$-dimensional case.

We introduce a number of notations. Let $x(t) = (x^1(t), x^2(t), \dots, x^n(t))$ be determined by equation (4). By $G(t)$ we denote the matrix $c(t, x(t))$, where $c(t,x) = b(t,x)b'(t,x)$; here $b'(t,x)$ is the matrix transposed to the matrix $b(t,x)$. Further let

$$A(t) = \left\{ \frac{\partial a^i(t,x)}{\partial x^j}\Big|_{x=x(t)} \right\}_{i,j=1}^n$$

and $G(t) = \{g^i_j(t)\}^n_{i,j=1}$ be the matrix of second moments of the process $x_1(t,\omega)$ (see formula (3)). It can be proved that

$$\dot{G}(t) = A(t)G(t) + G(t)A(t) + C(t), \tag{5}$$

where

$$\dot{G}(t) = \left\{ \frac{d}{dt} g^i_j(t) \right\}^n_{i,j=1}$$

and $g^i_j(0) = 0$. It is also easy to show that the rank of the matrix $G(t)$ is greater than or equal to the rank of the matrix $C(t)$. For example, if in equation (5) $A(t) = 0$, $n = 2$, and the rank of the matrix $G(s)$ for $s \in [t-\delta, t]$ is equal to 1, then we have

$$C(s) = \begin{pmatrix} c(s) & \lambda(s)c(s) \\ \lambda(s)c(s) & \lambda^2(s)c(s) \end{pmatrix},$$

where $c(s) > 0$ for $s \in [t-\delta, t]$, and if, moreover, $\lambda(s) \neq \text{const}$ for $s \in [t-\delta, t]$, then the rank of the matrix $G(s)$ is equal to 2, i.e. $\det[G(s)] \neq 0$ for $s \in (t-\delta, t]$.

**Theorem 1.** *In order that there exist such a nondegenerate matrix $Q(t)$ that the density $q^\varepsilon(t,x)$ of the distribution of the process*

$$y(t,\varepsilon,\omega) = \frac{1}{\varepsilon} Q(t)[x(t,\varepsilon,\omega) - x(t)]$$

*have the form*

$$q^\varepsilon(t,x) = \varphi_0(x) + \theta^\varepsilon(x,t), \qquad |\theta^\varepsilon(t,x)| \le K\varepsilon \ln^2 \frac{1}{\varepsilon}, \tag{6}$$

*where $K < \infty$ uniformly in $(x,\varepsilon)$ and*

$$\varphi_0(x) = (2\pi)^{-n/2} \exp\left(-\frac{|x|^2}{2}\right),$$

*it is necessary and sufficient that* $\det[G(t)] \neq 0$.

Now suppose that for some $t > 0$, $\det[G,t] = 0$ and the rank of the matrix $G(t)$ is $r > 0$. Then, without loss of generality, one may assume that for this $t$

$$x^i_1(t,\omega) = \sum_{j=1}^r d^i_j x^j_1(t,\omega) \quad \text{for } i = r+1, r+2, \dots, n.$$

**Theorem 2.** *Let the determinant of the matrix of second moments of the $(n+r)$-dimensional process*

$$(x_1^1(t,\omega), x_1^2(t,\omega), \dots, x_1^r(t,\omega), x_2^1(t,\omega), x_2^2(t,\omega), \dots, x_2^n(t,\omega))$$

*be different from zero. Then there exists such a matrix $\widetilde{Q}(t)$ that the density $\tilde{q}^\varepsilon(t,x)$ of the distribution of the process*

$$\tilde{y}(t,\varepsilon) = \widetilde{Q}(t) R(t,\varepsilon)[x(t,\varepsilon,\omega) - x(t)],$$

where

$$R(t,\varepsilon) = \begin{pmatrix} \frac{1}{\varepsilon} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\varepsilon} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\varepsilon} & 0 & \dots & 0 \\ \frac{1}{\varepsilon} d_1^{r+1} & \frac{1}{\varepsilon} d_2^{r+1} & \dots & \frac{1}{\varepsilon} d_r^{r+1} & \frac{1}{\varepsilon^2} & \dots & 0 \\ \frac{1}{\varepsilon} d_1^{n} & \frac{1}{\varepsilon} d_2^{n} & \dots & \frac{1}{\varepsilon} d_r^{n} & 0 & \dots & \frac{1}{\varepsilon^2} \end{pmatrix},$$

has the form

$$\tilde{q}^\varepsilon(t,x) = \widetilde{\varphi}_0(t,x) + \tilde{\theta}^\varepsilon(t,x), \qquad \left|\tilde{\theta}^\varepsilon(t,x)\right| \leqslant K\varepsilon \ln^3 \frac{1}{\varepsilon}, \tag{7}$$

where $K < \infty$ uniformly in $x, \varepsilon$, and $\widetilde{\varphi}_0(t,x)$ is an $n$-dimensional density of a certain nondegenerate distribution with zero means and the identity matrix of second moments.

For $\widetilde{\varphi}_0(t,x)$ one can write an explicit formula in terms of $\pi_2(t,x)$, the density of the distribution of the process $x_2(t,\omega)$.

If, however, the rank of the matrix $G(t)$ at the instant $t > 0$ is equal to zero, then it is easy to verify that $x_1(t,\omega) = 0$. Consequently,

$$\mathbf{M}|x_1(t,\omega)|^2 = \int_0^t \left( \sum_{i,j=1}^n |b_j^i(u, x(u))|^2 \right) du = 0,$$

and, hence, $b_j^i(u, x(u)) = 0$, $i,j = 1,2,\dots,n$, for $u \leqslant t$. From this we obtain that for $u \leqslant t$ the vector function $x(u)$ is a solution of equation (1) and, by uniqueness of the solution of equation (1), coincides with $x(u,\varepsilon,\omega)$, i.e. $x(u,\varepsilon,\omega) = x(u)$ for $u \leqslant t$, and

$$p^\varepsilon(t, x_0, x) = \delta(x - x(u)) \qquad \left(\delta(x-y) = 0 \text{ for } x \neq y \text{ and } \int_{R^n} \delta(x-y)\,dx = 1\right).$$

Theorems 1 and 2 can be strengthened by writing out the following terms of the asymptotic expansion in $\varepsilon$ for $p^\varepsilon(t, x_0, x)$. For example, the following theorem is valid:

**Theorem 3.** Let the matrix $G(t)$ have rank $r > 0$, and let the joint distribution of the processes

$$(x_1^1(t,\omega), x_1^2(t,\omega), \dots, x_1^r(t,\omega), x_2^1(t,\omega), x_2^2(t,\omega), \dots, \dots, x_m^n(t,\omega))$$

be nondegenerate (i.e., the determinant of the corresponding matrix of second moments is different from zero). Then

$$p^\varepsilon(t, x_0, x) = \frac{\psi_0^\varepsilon(t, x_0, x)}{\varepsilon^{2n-r}} + \frac{\psi_1^\varepsilon(t, x_0, x)}{\varepsilon^{2n-r-1}} + \dots + \frac{\psi_{m-2}^\varepsilon(t, x_0, x)}{\varepsilon^{2n-r-m+2}} + \frac{\widetilde{\psi}_{m-1}^\varepsilon(t, x_0, x)}{\varepsilon^{2n-r-m+1}}, \tag{8}$$

where

$$\left|\widetilde{\psi}_{m-1}^\varepsilon(t, x_0, x)\right| \leqslant K_{m-1} \ln^{m+1} \frac{1}{\varepsilon}, \qquad |\psi_i^\varepsilon(t, x_0, x)| \leqslant K_i, \quad i = 0, 1, \dots, m-2,$$

and $K_j < \infty,\ j = 0, 1, \dots, m-1$, uniformly in $x, \varepsilon$.

For the functions $\psi_i^\varepsilon(t, x_0, x)$, $i = 0, 1, \dots, m-2$, one can obtain explicit formulas in terms of $\pi_2(t,x), \pi_3(t,x), \dots, \pi_m(t,x)$, the densities of the distributions of the processes $x_2(t,\omega), x_3(t,\omega), \dots, x_m(t,\omega)$. Under the assumptions made by us, $p^\varepsilon(t, x_0, x)$ is the Green' s function for the equation

$$\frac{\partial U}{\partial t} = \frac{\varepsilon^2}{2} \sum_{i,j=1}^n c_j^i(t,x) \frac{\partial^2 U}{\partial x^i \partial x^j} + \sum_{i=1}^n a^i(t,x) \frac{\partial U}{\partial x^i}. \tag{9}$$

Knowing the asymptotics (8) for the Green' s function as $\varepsilon \to 0$, we can study the behavior of the solution of the Cauchy problem for equation (9), generally speaking, with an arbitrary initial function. For example, in the article by E. K. Isakova (3) the phenomenon of an "internal boundary layer" was studied. One of the essential assumptions in (3) is the nondegeneracy of the matrix $c(t,x)$. Using Theorem 3

one can prove analogous results also in the degenerate case. Moreover, the asymptotics of the solution of equation (9) as $\varepsilon \to 0$ will be determined by the rank of the matrix $G(t)$, which is related to the matrix $c(t, x(t))$ by equation (5).

To obtain uniform estimates in the asymptotic formulas (6), (7), and (8), the inequalities (see ([2]), Lemma 2)

$$\mathbf{M}\left\{\sup_{0\le t\le T} |x_k(t,\omega)|^l\right\} \le G_{k,l} < \infty, \quad k = 1, 2, \dots, m; \quad l = 1, 2, \dots, \tag{10}$$

where $x_k(t, \omega)$ are the terms of the asymptotic expansion (3) for the process $x(t, \varepsilon, \omega)$, proved insufficient. For this purpose certain theorems were needed which are also of independent interest.

**Theorem 4.** There exists $a_k > 0$ such that

$$\mathbf{M}\exp\left\{a_k \sup_{0\le t\le T} |x_k(t,\omega)|^{\frac{2}{k}}\right\} < \infty. \tag{11}$$

For any $\chi > 0$, $\mu > 0$, and some $t \in [0, T]$, there exist functions $a(u, x)$ and $b(u, x)$, with bounded partial derivatives with respect to $x$ up to order $(m + 1)$ inclusive, such that

$$\mathbf{M}\exp\left\{\mu |x_k(t,\omega)|^{\frac{2}{k}+\chi}\right\} = \infty. \tag{12}$$

**Theorem 5.** There exists $\gamma_{m+1} > 0$ such that

$$\mathbf{M}\exp\left\{\gamma_{m+1} \sup_{0\le t\le T} |x_{m+1}(t,\varepsilon,\omega)|^{\frac{1}{m+1}}\right\} \le K, \quad K < \infty \quad \text{uniformly in } \varepsilon. \tag{13}$$

The idea of the proof of inequalities (11) and (13) consists in using the stochastic integral equations written out in ([2]) for the process

$$\tilde{x}_m(t,\omega) = (x_1^1(t,\omega), x_1^2(t,\omega), \dots, x_1^n(t,\omega), x_2^1(t,\omega), \dots, x_m^n(t,\omega)),$$

one of K. Itô’ s formulas for stochastic integrals ([4]), and the following general assertion:

**Theorem 6.** Let $\mathfrak{F}_t$ be the $\sigma$-algebra generated by the sets

$$\{\omega : \xi(u_1,\omega) \in \Gamma_1, \dots, \xi(u_N,\omega) \in \Gamma_N;\ u_i \in [0,t],\ i = 1, 2, \dots, N\},$$

where $\Gamma_i$ are Borel subsets of $\mathbb{R}$ and $\xi(u, \omega)$ is a one-dimensional Wiener process. Suppose further that $\psi(t, \omega)$ is an $\mathfrak{F}_t$-measurable function of $\omega$ for each $t$ and is such that, for some $\chi > 0$,

$$\mathbf{M}\exp\left\{\sup_{t\in[0,T]} |\psi(t,\omega)|^{2\chi}\right\} < \infty.$$

Then for the random functions

$$\psi_1(t,\omega) = \int_0^t |\psi_1(t,\omega)|^\mu \, du, \qquad \psi_2(t,\omega) = \int_0^t |\psi(u,\omega)|^\mu \, d\xi(u,\omega)$$

the following inequalities hold:

$$\mathbf{M} \exp\left\{a_1 \sup_{0\le t\le T} |\psi_1(t,\omega)|^{\frac{2\chi}{\mu}}\right\} < \infty; \tag{14}$$

$$\mathbf{M} \exp\left\{a_2 \sup_{0\le t\le T} |\psi_2(t,\omega)|^{\frac{2\chi}{\chi+\mu}}\right\} < \infty \tag{15}$$

for some $a_1 > 0$, $a_2 > 0$.

Let us also note that one can give nontrivial examples in which the powers $\frac{2\chi}{\mu}$ in (14) and $\frac{2\chi}{\chi+\mu}$ in (15) cannot be increased by any number $\lambda > 0$.

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