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ON ONE METHOD FOR DETERMINING THE CONSTANTS OF THE CHRISTOFFEL-SCHWARZ INTEGRAL

![Fig. 1](figure)

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Abstract

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MATHEMATICS

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ON ONE METHOD FOR DETERMINING THE CONSTANTS OF THE CHRISTOFFEL-SCHWARZ INTEGRAL

(Presented by Academician M. A. Lavrent' ev on February 6, 1961)

The problem of the constants of the Christoffel-Schwarz integral was posed more than 90 years ago in the well-known works of E. Christoffel and H. Schwarz (^{1,2}). However, up to the present time there has been no sufficiently general and simple method for determining them. In the present article, power series are used to determine these constants, which substantially simplifies all computations.

Fig. 1

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1°. An arbitrary simply connected $(\mu+1)$-gon z, under the normalization indicated in Fig. 1, is mapped onto the half-plane ζ by means of the Christoffel-Schwarz integral:

$$z = D_1 \int \zeta^{\alpha_1-1}(1-\zeta)^{\alpha_2-1}\dots(1-k_\mu\zeta)^{\alpha_\mu-1}\,d\zeta + D_2, \tag{1}$$

whose constants $k_3, k_4, \dots, k_\mu, D_1, D_2$ must be determined in advance.

For computing the improper integrals I_i needed below, we use the formula which we obtain by expanding each factor of the integrand into a binomial series:

$$I_i = \int_{1/k_i}^{1/k_j} \zeta^{\nu_i+\beta_i-1}\left(1-\frac{1}{k_i\zeta}\right)^{\alpha_i-1}(1-k_j\zeta)^{\alpha_j-1}\,d\zeta =$$

$$= -\frac{\sin\pi\beta_{i-1}}{\sin\pi\beta_i}\,k_i^{-\nu_i-\beta_i}\sum_{m=0}^{\infty} b^{(i-1)}_{\nu_i+m}A^{(m)}_{j/i} + k_j^{-\nu_i-\beta_i}\sum_{m=0}^{\infty} b^{(i)}_{\nu_i-m}A^{(1)}_{j/i}, \tag{2}$$

$$i = 1,2,3,\dots; \qquad j = i+1; \qquad \beta_1 = \alpha_1; \qquad \beta_n = \beta_{n-1}+\alpha_n-1; \tag{3}$$

$$b_0^{(i)} = \frac{\Gamma(\beta_i)\Gamma(\alpha_j)}{\Gamma(\beta_i+\alpha_j)}; \qquad \frac{b^{(i)}_{\nu+1}}{b^{(i)}_{\nu}} = \frac{\nu+\beta_i}{\nu+1+\beta_j}; \qquad \frac{b^{(i)}_{-\nu-1}}{b^{(i)}_{-\nu}} = \frac{\nu-\beta_j}{\nu+1-\beta_i}; \tag{4}$$

$$A^{(m)}_{j/i} = \alpha^{(\nu)}_m \left(\frac{k_j}{k_i}\right)^m; \qquad \alpha^{(\nu)}_0 = 1; \qquad \frac{\alpha^{(\nu)}_{m+1}}{\alpha^{(\nu)}_m} = \frac{m+1-\alpha_\nu}{m+1}. \tag{5}$$

For $\beta_i = 0, \pm 1, ...$, formula (2) loses its meaning, but in this case as well it is easy to obtain an analogous formula whose composition will include a logarithmic function.

According to formula (1) we have

$$\begin{aligned}
\frac{l_i}{|D_1|} &= \int_{1/k_i}^{1/k_j} \xi^{\alpha_1-1}(\xi-1)^{\alpha_2-1} \ldots (k_i\xi-1)^{\alpha_i-1}(1-k_j\xi)^{\alpha_j-1} \ldots (1-k_\mu\xi)^{\alpha_\mu-1}\,d\xi \\
&= k_3^{\alpha_3-1} \ldots k_i^{\alpha_i-1} \int_{1/k_i}^{1/k_j} \xi^{\beta_i-1}\left(1-\frac{1}{\xi}\right)^{\alpha_2-1} \ldots \left(1-\frac{1}{k_i\xi}\right)^{\alpha_i-1}(1-k_j\xi)^{\alpha_j-1} \ldots \\
&\qquad \ldots (1-k_\mu\xi)^{\alpha_\mu-1}\,d\xi.
\end{aligned}$$

Expanding, in the integrand thus obtained, all the brackets except the i-th and j-th into binomial series and then using formula (2), we obtain a system of equations for determining the required constants of the Christoffel-Schwarz integral:

$$\begin{aligned}
\frac{M\lambda_1}{b_0^{(1)}} &= \sum_{p,\ldots,m=0}^{\infty} A^{(p)}_{\mu\,\frac{\mu}{2}} \ldots A^{(n)}_{4\,\frac{4}{2}} A^{(m)}_{3\,\frac{3}{2}} \bar{b}^{(1)}_{p+\ldots+n+m}; \\
\frac{M\lambda_2}{b_0^{(2)}} &= k_3^{-\beta_2} \sum_{p,\ldots,m=0}^{\infty} A^{(p)}_{\mu\,\frac{\mu}{3}} \ldots A^{(n)}_{4\,\frac{4}{3}} A^{(m)}_{2\,\frac{3}{2}} \bar{b}^{(2)}_{p+\ldots+n-m}; \\
\frac{M\lambda_3}{b_0^{(3)}} &= k_3^{\alpha_3-1} - k_4^{-\beta_3} \sum_{p,\ldots,m=0}^{\infty} A^{(p)}_{\mu\,\frac{\mu}{4}} \ldots A^{(n)}_{2\,\frac{4}{2}} A^{(m)}_{3\,\frac{4}{3}} \bar{b}^{(3)}_{p+\ldots-n-m}; \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{M\lambda_{\mu-1}}{b_0^{(\mu-1)}} &= k_3^{\alpha_3-1} k_4^{\alpha_4-1} \ldots k_\mu^{-\beta_{\mu-1}} \sum_{p,\ldots,m=0}^{\infty} A^{(p)}_{2\,\frac{\mu}{2}} \ldots A^{(n)}_{\mu-2\,\frac{\mu}{\mu-2}} A^{m}_{\mu-1\,\frac{\mu}{\mu-1}} \bar{b}^{(\mu-1)}_{-p-\ldots-n-m},
\end{aligned} \tag{6}$$

where

$$\lambda_1 = l_1; \qquad \lambda_\nu = \frac{1}{\sin\pi\beta_\nu}\sum_{j=1}^{\nu} l_j \sin\pi\beta_j; \qquad M = \frac{1}{|D_1|}; \qquad \bar{b}^{(i)}_\nu = \frac{b^{(i)}_\nu}{b^{(i)}_0}. \tag{7}$$

The coefficients $\bar{b}$ are computed by the same recurrence formulas (4), but under the condition that $\bar{b}_0^{(i)} = 1$. All sums in the system (6) have multiplicity equal to $\mu - 2$.

Eliminating M from equations (6), we determine the constants $k_3, k_4, \dots, k_\mu$ by the Newton–Fourier method. The first approximations are conveniently found by the iteration method.

2°. Let us consider in greater detail the case of a quadrilateral ($\mu = 3$) and of a pentagon ($\mu = 4$).

For $\mu = 3$, denoting $k_3 = k$ and taking into account that $k_2 = 1$, we have from (5):

$$A_{3/2}^{(m)} = \alpha_m^{(3)} k^m; \qquad A_{2\,3/2}^{(m)} = \alpha_m^{(2)} k^m.$$

Introducing next the new notation

$$A_m = \alpha_m^{(3)} \bar{b}_m^{(1)}; \qquad B_m = \alpha_m^{(2)} \bar{b}_m^{(2)} \tag{8}$$

and taking into account formulas (4), (5), (7), from the system (6) for $\mu = 3$ we obtain

$$\frac{Ml_1}{b_0^{(1)}} = \sum_{m=0}^{\infty} A_m k^m; \qquad A_0 = 1; \qquad \frac{A_{m+1}}{A_m} = \frac{(m+1-\alpha_3)(m+\alpha_1)}{(m+1)(m+\alpha_1+\alpha_2)}; \tag{9}$$

$$\frac{M\lambda_2}{b_0^{(2)}} = k^{-\beta_2} \sum_{m=0}^{\infty} B_m k^m; \qquad B_2 = 1; \qquad \frac{B_{m+1}}{B_m} = \frac{(m+1-\alpha_2)(m+\alpha_4)}{(m+1)(m+\alpha_3+\alpha_4)}. \tag{10}$$

Dividing now (10) by (9), we find an equation for determining k:

$$f(k) = g - k^{\beta_2} I_c = 0, \tag{11}$$

where

$$g = \frac{l_1 b_0^{(2)}}{\lambda_2 b_0^{(1)}} = \frac{l_1 \Gamma(\alpha_1+\alpha_2)\Gamma(\beta_2)\Gamma(\alpha_3)}{\lambda_2 \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2+\alpha_3)}; \qquad I_c = \frac{\sum A_m k^m}{\sum B_m k^m} = \sum_{m=0}^{\infty} C_m k^m. \tag{12}$$

The coefficients of the series I_c are determined from the recurrence formulas $((3^3), \S 20)$

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$$C_0 = 1; \qquad C_m = A_m - (B_m + C_1 B_{m-1} + \cdots + C_{m-1} B_1). \tag{13}$$

Putting $I_c = 1$ in (11), we obtain the zeroth approximation for k:

$$k_0 = g^{1/\beta_2}; \qquad \beta_2 = \alpha_1 + \alpha_2 - 1; \qquad \lambda_2 = l_2 + l_1 \frac{\sin \pi\alpha_1}{\sin \pi\beta_2}. \tag{14}$$

More accurate results are given by the first-approximation formula:

$$k_1 = \frac{k_0}{1 - d_2 k_0}; \qquad d_2 = -\frac{C_1}{\beta_2} = \frac{\alpha_1\alpha_3 + \alpha_2\alpha_4}{\beta_2^2 - 1}. \tag{15}$$

Further refinement is carried out by Newton' s formula

$$k_{n+1} = k_n - \frac{f(k_n)}{f'(k_n)}; \qquad f'(k) = -k^{\beta_2 - 1}\left(\beta_2 I_c + \sum_{m=1}^{\infty} m C_m k^m\right). \tag{16}$$

Having determined the constant k, we find the constants D_1 and D_2 from equations (9) and (1) in the usual way ([1]).

We shall now give, for the constant k, an expansion in a series in powers of the approximations k_0 and k_1. According to equations (12), (13), (15) we have

$$g = k^{\beta_2}\{1 + C_1 k + C_2 k^2 + \cdots\}.$$

Reverting the resulting series $((3^6), \S 73)$, we solve the problem:

$$k = k_0 + d_2 k_0^2 + d_3 k_0^3 + \cdots; \qquad k_0 = g^{1/\beta_2},$$

$$d_2 = -\frac{C_1}{\beta_2}; \qquad d_3 = \frac{1}{\beta_2}\left(\frac{3 + \beta_2}{2\beta_2} C_1^2 - C_2\right). \tag{17}$$

Assuming approximately that $d_n \approx d_2^{n-1}$, we obtain from (17) the first-approximation formula (15), after which, regrouping the series (17) in powers of k_1, we have:

$$k = k_1 + a_3 k_1^3 + a_4 k_1^4 + \cdots; \qquad a_3 = d_2^2 - d_3; \qquad a_4 = d_4 - d_2^3 - 3a_3 d_2. \tag{18}$$

In a similar way we obtain two formulas of the second approximation:

$$k_2 = k_1(1 + a_3 k_1^2); \qquad k_{II} = k_0 + \frac{d_2 k_0^2}{1 - d_{32} k_0}; \qquad d_{32} = \frac{d_3}{d_2}. \tag{19}$$

Table 1

No.	α_1	α_2	α_3	α_4	l_2 : $l_1; h$: l_1	k_{exact}	k_{pr}	k_{II}	k_2
1	0,20	0,55	0,60	0,65	0,600000	0,0552759	0,0552762	0,0552759	0,0552764
2	0,20	0,55	0,60	0,65	0,500000	0,2126450	0,21267	0,21261	0,21273
3	0,20	0,55	0,60	0,65	0,3988633	0,5000000	0,5007	0,4983	0,5031
4	0,55	0,60	0,65	0,20	0,4216054	0,2042829	0,20430	0,20426	0,20435
5	0,55	0,60	0,65	0,20	0,5007658	0,0375183	0,0375183	0,0375183	0,0375183
6	0,30	1,20	0,75	−0,25	1,500000	0,1133944	0,1133945	0,1133945	0,1133946
7	0,30	1,20	0,75	−0,25	1,000000	0,1860865	0,1860878	0,1860876	0,1860876
8	0,30	1,20	0,75	−0,25	0,9371541	0,2000000	0,2000016	0,2000014	0,2000018
9	0,30	1,20	0,75	−0,25	0,500000	0,3616651	0,361687	0,361685	0,361689
10	0,30	1,20	0,75	−0,25	0,3149219	0,5000000	0,50011	0,500098	0,500113
11	0,28	1,89	0	−1,17	5,000000	0,2215004	0,2214988	0,2214458	0,2215518
12	0,28	1,89	0	−1,17	3,000000	0,3092554	0,309238	0,309013	0,309462
13	0,28	1,89	0	−1,17	2,000000	0,3920668	0,39198	0,39133	0,39263
14	0,28	1,89	0	−1,17	1,2388590	0,5000000	0,4996	0,4975	0,5016
15	0,28	1,89	0	−1,17	0,6416990	0,6500000	0,6472	0,6389	0,6555
16	0,51	0,24	1,15	0,10	1,000000	0,0692268	0,0692270	0,0692266	0,0692276

Table 1 gives results for 16 examples of quadrilaterals, closed, open, and degenerate ($\alpha_3 = 0$). The exact value k_{exact} was determined by formula (16) with 7 decimal places, and the approximate $k_{\text{pr}} = 0.5(k_{II} + k_2)$. Under numbers 1–5 and 11–15 in Fig. 2, graphs are constructed for one-parameter families of the corresponding quadrilaterals. If $k > 0.5$, it is better to compute the additional constant $k_* = 1 - k$, for which one must pass to the transposed quadrilateral: $l_1^* = l_2$; $l_2^* = l_3$; $\alpha_2^* = \alpha_{j+1}$; $\alpha_4^* = \alpha_1$. For more detail on the quadrilateral, see (^{5}).

In the case of a pentagon ($\mu = 4$), taking into account that $k_2 = 1$, it is more convenient to use the simpler notation

$$A_{ij}^{(\nu)} = \alpha_i^{(\nu)} k_j^{\nu}; \qquad A_{i\tau}^{(\nu)} = \alpha_i^{(\nu)} \tau^{\nu}; \quad \tau = k_4/k_3. \tag{20}$$

Then, starting from (4)–(7), we finally arrive at a system of two equations for determining the constants k_3, k_4:

Fig. 2

$$f(k_3, k_4) = g_1 I_{2\tau} - k_3^{\beta_2} I_{14} = 0; \qquad \varphi(k_3, k_4) = g_3 I_{2\tau} - \tau^{-\beta_3} I_{34} = 0;$$

$$I_{14} = \sum_{n=0}^{\infty} S_{133}^{(n)} A_{44}^{(n)}; \qquad I_{2\tau} = \sum_{n=0}^{\infty} S_{223}^{(n)} A_{4\tau}^{(n)}; \qquad I_{34} = \sum_{n=0}^{\infty} S_{33\tau}^{(n)} A_{24}^{(n)};$$

$$S_{133}^{(n)} = \sum_{m=0}^{\infty} \bar{b}_{n+m}^{(1)} A_{33}^{(m)}; \qquad S_{223}^{(n)} = \sum_{m=0}^{\infty} \bar{b}_{n-m}^{(2)} A_{23}^{(n)}; \qquad S_{33\tau}^{(n)} = \sum_{m=0}^{\infty} \bar{b}_{-n-m}^{(3)} A_{3\tau}^{(m)};$$

$$g_1 = \frac{\lambda_1 \Gamma(\beta_2)\Gamma(\alpha_3)\Gamma(\alpha_1 + \alpha_3)}{\lambda_2 \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2 + \alpha_2)}; \qquad g_3 = \frac{\lambda_3 \Gamma(\beta_3)\Gamma(\alpha_3)\Gamma(\beta_3 + \alpha_4)}{\lambda_2 \Gamma(\beta_3)\Gamma(\alpha_4)\Gamma(\beta_2 + \alpha_3)}. \tag{21}$$

We solve system (21) by the Newton—Fourier method; the initial values for it are found by the method of iterations:

$$k_3 = \left\{ g_1 \frac{I_{2\tau}}{I_{14}} \right\}^{1/\beta_2}; \qquad \tau = \left\{ g_3 \frac{I_{2\tau}}{I_{34}} \right\}^{-1/\beta_3}; \qquad k_3^{(0)} = g_1^{1/\beta_2}; \qquad \tau^{(0)} = g_3^{-1/\beta_3}. \tag{22}$$

Table 2

No.	α_1	α_2	α_3	α_4	α_5	$l_2 : l_1$	$l_3 : l_1$	k_3	k_4
1	0.70	0.55	0.60	0.65	0.50	1.25	1.60	0.458884	0.306130500
2	0.30	1.20	0.25	1.40	–0.15	1.75	2.00	0.329684	0.208852677

Table 2 gives the results for two pentagons, a closed and an open one. In the general case, for $\mu \geq 5$, initial values with 2—3 significant figures can easily be determined by electric modeling (^{4}).

For regular $(\mu + 1)$-gons we have:

$$1 - k_3 = k_3(1 - k_4) = k_4(1 - k_5) = \ldots = k_{\mu-1}(1 - k_\mu) = k_\mu. \tag{23}$$

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