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Abstract

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MATHEMATICS

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ERROR ESTIMATE FOR THE NUMERICAL SOLUTION OF A LINEAR INTEGRAL EQUATION

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Consider the Fredholm integral equation of the second kind

$$\varphi(s) = \lambda \int_a^b K(s,t)\varphi(t)\,dt + f(s), \tag{1}$$

where $f(s)$ and $K(s,t)$ are continuous functions on the interval $a \le s \le b$ and in the square $a \le s,\ t \le b$, respectively. Suppose that a table is given

$$\begin{array}{c|c} t_1 & \Phi_1 \\ t_2 & \Phi_2 \\ \vdots & \vdots \\ t_n & \Phi_n \end{array} \tag{2}$$

with

$$a \le t_1 < t_2 < \dots < t_n \le b,$$

which represents an approximate numerical solution of equation (1). The method by which table (2) is obtained is immaterial.

Denote by φ_i the value of the solution $\varphi(s)$ of equation (1) at the point t_i: $\varphi_i = \varphi(t_i),\ i = 1, 2, \dots, n$. Below an estimate is given for the norm of the difference $\varphi - \Phi$ of the vectors

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n), \qquad \Phi = (\Phi_1, \Phi_2, \dots, \Phi_n).$$

By the norm of a vector $x = (\xi_1, \xi_2, \dots, \xi_n)$ we shall mean

$$\|x\|_l = \max_i |\xi_i|.$$

By the norm of a function $x(s)$ continuous on $[a, b]$ we shall mean the norm in the space C of continuous functions,

$$\|x\|_C = \max_{a \le s \le b} |x(s)|.$$

Take the quadrature formula

$$\int_a^b F(t)\,dt = \sum_{j=1}^n A_j F(t_j), \tag{3}$$

constructed using the nodes $t_1, t_2, \dots, t_n$, the arguments of table (2). It is known that such formulas exist.

In what follows an important role is played by the functions (see ([1]))

$$\varepsilon(s,t) = \int_a^b K(s,\tau)K(\tau,t)\,d\tau - \sum_{j=1}^n A_j K(s,t_j)K(t_j,t); \tag{4}$$

$$\varepsilon_f(s) = \int_a^b K(s,\tau)f(\tau)\,d\tau - \sum_{j=1}^n A_j K(s,t_j)f(t_j). \tag{5}$$

The function $\varepsilon(s,t)$ is the quadrature error arising in the computation of the integral with respect to τ of the function $K(s,\tau)K(\tau,t)$ by means of the quadrature formula (3); $\varepsilon_f(s)$ is the quadrature error of the function $K(s,t)f(t)$.

We shall use the notation

$$\varepsilon_1 = \max_{a \le s \le b} \sum_{j=1}^n \left|A_j \varepsilon(s,t_j)\right|; \tag{6}$$

$$\varepsilon = \max_{1 \le i \le n} \int_a^b |\varepsilon(t_i,t)|\,dt; \tag{7}$$

$$K_1 = \max_{a \le s \le b} \sum_{j=1}^n \left|A_{jK}(s,t_j)\right|. \tag{8}$$

If $\psi(t)$ is a certain function given on $[a, b]$, then the vector with components $\psi_i = \psi(t_i)$ will be denoted by ψ.

Theorem. Let λ be a proper value of equation (1), let an estimate of the norm in C of the resolvent $R(s,t,\lambda)$ of the kernel $K(s,t)$ be known,

$$\max_{a\le s\le b}\int_a^b |R(s,t,\lambda)|\,dt \le \Gamma \tag{9}$$

and let the inequality

$$q = |\lambda|^2\varepsilon_1 B < 1, \tag{10}$$

be satisfied, where

$$B = 1 + |\lambda|\Gamma \tag{11}$$

and ε_1 is defined by equality (6). Then the estimate

$$\|\varphi - \Phi\|_l \le \frac{1+|\lambda|K_1 B}{1-q}\left(|\lambda|\|\varepsilon_f\|_l + |\lambda|^2 B\|f\|_C\varepsilon + \|\rho\|_l\right). \tag{12}$$

is valid. Here the vector ρ is defined by the equality

$$\rho = (I - \lambda L)\Phi - f, \tag{13}$$

where L is the matrix of order n, generated by the kernel $K(s,t)$ and the quadrature formula (3):

$$L = (A_{jK_{ij}}), \qquad K_{ij} = K(t_i, t_j). \tag{14}$$

Proof. The following equality holds (see (1)):

$$\varphi(s) = \lambda\sum_{j=1}^{n} A_{jK}(s,t_j)\varphi_j + f(s) + \lambda\alpha(s), \tag{15}$$

which follows from the integral equation (1). Here

$$\alpha(s) = \varepsilon_f(s) + \lambda\int_a^b \varepsilon(s,t)\varphi(t)\,dt \tag{16}$$

and $\varphi(t)$ is the solution of equation (1). Put in (15) $s = t_1, t_2, \dots, t_n$. We obtain the vector equality

$$(I - \lambda L)\varphi = f + \lambda\alpha. \tag{17}$$

Adding the left- and right-hand sides of equalities (13) and (17), we obtain

$$(I - \lambda L)(\varphi - \Phi) = \lambda\alpha - \rho. \tag{18}$$

The matrix on the left-hand side of (18) has an inverse. This follows from (10) and the following matrix equality:

$$(I - \lambda L)^{-1} = (I + \lambda^2 G^{-1})(I + \lambda R), \tag{19}$$

where G and R are matrices of order n:

$$G = \left(A_j \left[\varepsilon(t_i, t_j) + \lambda \int_a^b R(t_i, t, \lambda)\varepsilon(t, t_j)\, dt \right] \right),$$

$$R = (A_{jR}(t_i, t_j, \lambda)).$$

We do not give the proof of equality (19) here. From equality (19) and conditions (9) and (10) we obtain an estimate for the norm of the inverse matrix

$$\|(I - \lambda L)^{-1}\|_I \leq \frac{1 + |\lambda| K_1 B}{1 - q}. \tag{20}$$

Now from (18) and (20) we obtain

$$\|\varphi - \Phi\|_I \leq \frac{1 + |\lambda| K_1 B}{1 - q} (|\lambda| \|\alpha\|_I + \|\rho\|_I). \tag{21}$$

With the help of (16) we find

$$\|\alpha\|_I \leq \|\varepsilon_f\|_I + |\lambda| \|\varphi\|_C \varepsilon \leq \|\varepsilon_f\|_I + |\lambda| B \|f\|_C \varepsilon. \tag{22}$$

Here we have used the fact that the norm of the solution $\varphi(s)$ of equation (1), by virtue of (9) and (11), satisfies the inequality $\|\varphi\|_C \leq B\|f\|_C$. If in (21) one replaces $\|\alpha\|_I$ by the right-hand side of inequality (22), then we obtain (12). The theorem is proved.

Estimate (12) is a posteriori. The quadrature formula (3) may be regarded as a parameter whose choice is at one' s disposal. If table (2) is obtained by the method of mechanical quadratures with the help of formula (3), then the vector Q, defined by equality (13), is equal to zero. In this case $\|\rho\|_I = 0$, and (12) represents an a priori error estimate for the numerical solution obtained by the method of mechanical quadratures.

Example. We indicate an error estimate arising in solving the integral equation

$$\varphi(s) = 0.5 \int_0^1 \frac{5}{13 - 12\cos 2\pi(s+t)} \varphi(t)\, dt + 1 \tag{23}$$

by the method of mechanical quadratures using the left-rectangle formula for $n = 10$:

$$t_j = (j-1)0.1; \quad A_j = 0.1; \quad j = 1, 2, \dots, 10.$$

We have: $\lambda = 0.5$; $\|f\|_C = 1$; $\|\rho\|_I = 0$; $\Gamma = 2$; $B = 2$; $K_1 < 1.04$; $\varepsilon_1 < 0.2$; $\|\varepsilon_f\|_I < 0.036$; $\varepsilon = 0.036$. With the help of (12) we find $\|\varphi - \Phi\|_I < 0.082$.

In the example under consideration it is easy to indicate Φ, the solution of the system $(I - \lambda L)\Phi = f$, which turns out to be a vector with identical components equal to $2.0731\dots$. The exact solution of equation (23) is $\varphi(s) = 2$, so that $\|\varphi - \Phi\|_I = 0.073$. Thus the estimate exceeds the actual error by approximately a factor of 1.1.

The main part of the computations in the example, connected with determining upper bounds for the quantities $K_1, \varepsilon_1, \|\varepsilon_f\|_I$, and ε, was carried out by G. A. Domanovskii.

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Note: Figure translations are in progress. See original paper for figures.