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Abstract

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ON A CERTAIN CRITERION FOR THE COMPLETENESS OF A SYSTEM OF EIGEN AND ASSOCIATED VECTORS OF A POLYNOMIAL PENCIL OF OPERATORS

(Presented by Academician M. V. Keldysh, 28 VI 1961)

The present note is devoted to proving a proposition that generalizes one of the results of M. V. Keldysh ([1]).

Following the article ([2]), denote by $\mathfrak{S}_p$ $(p > 0)$ the class of all completely continuous operators A in a Hilbert space $\mathfrak{H}$ for which

$$|A|_p^p = \mathrm{Sp}((A^*A)^{p/2}) < \infty,$$

and by $\mathfrak{S}_\infty$ the class of all completely continuous operators. Recall that $\mathfrak{S}_p$ for $p \geq 1$ is a Banach space with norm $|A|_p$; moreover, $\mathfrak{S}_p$ $(p > 0)$ is a closed two-sided self-adjoint ideal in the ring $\mathfrak{R}$ of linear bounded operators.

I. Ts. Gokhberg and M. G. Krein communicated to the author a proof of the following theorem:

I. *The scheme of root subspaces of the operator $A = H(I + S)$ is complete in $\mathfrak{H}$, if the operator A is annihilated only at zero, $H = H^* \in \mathfrak{S}_\infty$, and for some $p \geq 1$ at least one of the operators HS or SH belongs to $\mathfrak{S}_p$.*

This proposition is equivalent to the following:

II. *The system of eigen and associated vectors of the pencil*

$$x = Tx + \lambda Hx$$

is complete in $\mathfrak{H}$, if the operator $H = H^* \in \mathfrak{S}_\infty$ is complete, $T \in \mathfrak{S}$, and for some $p \geq 1$ at least one of the operators HT or TH belongs to $\mathfrak{S}_p$.

Assertion II is a generalization of the completeness criterion, due to M. V. Keldysh ([1]), for the case $n = 1$. The proof of assertions I, II was obtained by I. Ts. Gokhberg and M. G. Krein as a result of developing the proof of M. V. Keldysh' s theorem for the case $n = 1$, communicated to them by V. B. Lidskii and based on ideas of M. V. Keldysh and Browder ([3]).

Here it will be shown that a modification of the arguments of these authors makes it possible to establish the following proposition.

Theorem. *The system of eigen and associated vectors of the pencil*

$$x = T_0 x + \lambda T_1 x + \dots + \lambda^{n-1} T_{n-1} x + \lambda^{nHx} \tag{1}$$

is n-fold complete in $\mathfrak{H}$, if the operator $H = H^* \in \mathfrak{S}_\infty$ is complete, $T_k = H^{k/n} B_k$, where $B_k \in \mathfrak{S}_\infty$ $(k = 0, 1, \dots, n-1)$, and for some $p \geq 1$ the operators $T_0 H$, $T_k \in \mathfrak{S}_p$ $(k = 1, 2, \dots, n-1)$.

Let us explain that by $H^{k/n}$ we mean $(H^{1/n})^k$, where $H^{1/n}$ is any normal operator whose n-th power is equal to H.

The theorem of M. V. Keldysh ([1]) (see also ([4])) differs from the one formulated above by the stronger requirement $H \in \mathfrak{S}_p$.

Proof. We rewrite the equation in the form of the system

$$\begin{aligned} x^0 &= T_0 x^0 + T_1 x^1 + \dots + T_{n-1} x^{n-1} + \lambda H x^{n-1}, \\ x^1 &= \lambda x^0, \\ x^2 &= \lambda x^1, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x^{n-1} &= \lambda x^{n-2}, \end{aligned} \tag{2}$$

where $x_0 = x$. Introducing the space $\widetilde{\mathfrak{H}}$, which is the orthogonal sum of n copies of the given $\mathfrak{H}$, one can represent system (2) in the form of the equation

$$\tilde{x} = \mathcal{T} \tilde{x} + \lambda \mathcal{H} \tilde{x}, \tag{3}$$

where $\tilde{x} = \begin{pmatrix} x^0 \\ \vdots \\ x^{n-1} \end{pmatrix} \in \widetilde{\mathfrak{H}}$, and the operators $\mathcal{T}$ and $\mathcal{H}$ in $\widetilde{\mathfrak{H}}$ are given by the operator matrices

$$\mathcal{T} = \begin{pmatrix} T_0 T_1 \dots T_{n-1} \\ 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} 0 & 0 & \dots & 0 & H \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & \dots & I & 0 \end{pmatrix}.$$

Without loss of generality one may assume that the operator $I - T_0$ is continuously invertible. Indeed, this condition can be achieved by the replacement $\lambda \to \lambda + a$, where $|a|$ is a sufficiently large number and $\arg a = \pi/2n$. We note that together with the operator $I - T_0$, the operator $I - \mathcal{T}$ will also be continuously invertible. The problem is reduced to proving completeness of the eigenvectors and associated vectors of the linear pencil (3), which is equivalent to completeness of the system of root subspaces of the weakly perturbed operator $\mathcal{A} = (I + \mathcal{S})\mathcal{H}$, where $I + \mathcal{S} = (I - \mathcal{T})^{-1}$.

Denote by $\widetilde{\mathfrak{H}}_0$ the orthogonal complement of the linear span of the root subspaces of the operator $\mathcal{A}$. Then $\widetilde{\mathfrak{H}}_0$ is invariant with respect to $\mathcal{A}^*$, and the operator $\mathcal{A}_1 = \mathcal{P}\mathcal{A}^*\mathcal{P}$, where $\mathcal{P}$ is the orthoprojection from $\widetilde{\mathfrak{H}}$ onto $\widetilde{\mathfrak{H}}_0$, is Volterra; consequently the operator-function $\Gamma_\zeta = (I - \zeta\mathcal{A}_1)^{-1}$ is entire.

Lemma 1. *As $\zeta \to \infty$ in the domain $G_\alpha^{(n)}$ $(0 < \alpha < \pi/2n)$, obtained from the complex plane by removing the $2n$ angles $|\arg z - \pi k/n| < \alpha$ $(k = 0, 1, \dots, 2n-1)$, the relation*

$$|\Gamma_\zeta| = O\left(|\zeta|^{2n-2}\right) \tag{4}$$

holds.

Relation (4) will be established if it turns out that

$$\left|(I - \mathcal{T} - \zeta\mathcal{H})^{-1}\right| = O\left(|\zeta|^{2n-2}\right).$$

But

$$(I - \mathcal{T} - \zeta\mathcal{H})^{-1} =$$

$$= \begin{pmatrix} R_\zeta & (T_1 + \zeta T_2 + \dots + \zeta^{n-2}T_{n-1} + \zeta^{n-1}H)\,R_\zeta & \dots & (T_{n-1} + \zeta H)R_\zeta \\ \zeta R_\zeta & I + \zeta(T_1 + \zeta T_2 + \dots + \zeta^{n-1}H)R_\zeta & \dots & \zeta(T^{n-1} + \zeta H)R_\zeta \\ \cdot & \cdot & \cdot & \cdot \\ \zeta^{n-1}R_\zeta & \zeta^{n-2}I + \zeta^{n-1}(T_1 + \zeta T_2 + \dots + \zeta^{n-1}H)R_\zeta & \dots & I + \zeta^{n-1}(T_{n-1} + \zeta H)R_\zeta \end{pmatrix},$$

where

$$R_\zeta = (I - T_0 - \zeta T_1 - \dots - \zeta^{n-1}T_{n-1} - \zeta^n H)^{-1}.$$

Since it is known [4] that $|R_\zeta| = O(1)$, (4) is proved.

Lemma 2. *The function Γ_ζ is of finite order.*

Indeed,

$$\mathcal{T}\mathcal{H} = \begin{pmatrix} T_1 & T_2 & \cdots & T_{n-1} & T_0 H \\ 0 & & & & \end{pmatrix} \in \mathfrak{S}_p$$

and $\mathcal{G}\mathcal{H} = (I - \mathcal{T})^{-1}\mathcal{T}\mathcal{H} \in \mathfrak{S}_p$. Therefore

$$\mathcal{A}_1^n = \mathcal{P}\mathcal{A}^{*n}\mathcal{P} = \mathcal{P}(\mathcal{H}^* + \mathcal{H}^*\mathcal{S}^*)^n\mathcal{P} = \mathcal{P}\mathcal{H}^{*n}\mathcal{P} + \mathcal{P}\mathcal{M}\mathcal{P}.$$

Since $H^{*n} = (\delta_{ik}H)_{i,k=1}^n$ is a self-adjoint operator and $\mathcal{M} \in \mathfrak{S}_p$, on the basis of the theorem on the relation between the Hermitian components of a Volterra operator ([5,6]), we conclude that $\mathcal{A}_1^n \in \mathfrak{S}_p$.

Therefore ([7,8]) the operator-function $(I - \zeta^n\mathcal{A}_1^n)^{-1}$, and together with it also

$$\Gamma_\zeta = (I + \zeta\mathcal{A}_1 + \cdots + \zeta^{n-1}\mathcal{A}_1^{n-1})(I - \zeta^n\mathcal{A}_1^n)^{-1}$$

turn out to be entire of order not exceeding $n([p]+1)$.

Applying the Phragmén–Lindelöf principle to the function Γ_ζ in the domain complementary to $G_\alpha^{(n)}$ $(\alpha < \pi/n(p+1))$, we obtain that $|\Gamma_\zeta| = O(|\zeta|^{2n-2})$ as $\zeta \to \infty$. Therefore the operator $\mathcal{A}_1$ turns out to be nilpotent, and hence there exist vectors on which

$$\mathcal{A}^* = \mathcal{H}^*(I + \mathcal{S}^*)$$

is annihilated, which is impossible. Thus, $\widetilde{\mathfrak{H}}_0 = 0$.

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Note: Figure translations are in progress. See original paper for figures.