

# RIEMANNIAN FIBERED SPACES WITH A ONE-PARAMETER HOLONOMY GROUP

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **RIEMANNIAN FIBERED SPACES WITH A ONE-PARAMETER HOLONOMY GROUP**

*(Presented by Academician P. S. Aleksandrov on 13 II 1961)*

**1. Definitions.** A Riemannian space of  $n + m$  dimensions with a positive-definite metric, admitting a fibration into  $m$ -dimensional totally geodesic and parallel submanifolds, is called a **Riemannian fibered space**  $V^{n+m}$ . The leaves are called **parallel** if, under parallel translation of a vector tangent to a leaf along any curve orthogonal at each of its points to the leaf passing through that point, the vector remains tangent to the leaf at every point of the curve.

Associating with each point of  $V^{n+m}$  an orthonormal frame, we obtain the following structure equations <sup>(1)</sup>:

$$D\omega_u = [\omega, \omega_{vu}], \quad \omega_{vu} + \omega_{uv} = 0,$$

$$t, u, v, w = 1, \dots, n + m$$

$$D\omega_{uv} = [\omega_{uv}, \omega_{uv}] + R_{uvwt}[\omega_w, \omega_t].$$

If the first  $m$  vectors of the frame at each point are chosen tangent to the leaf passing through that point, we obtain the following equations for the family of totally geodesic and parallel leaves

$$\omega_{ia} = l_{iab}\omega_b, \quad l_{iab} + l_{iba} = 0; \quad i = 1, \dots, m; \quad a, b = m + 1, \dots, m + n. \quad (1)$$

Differentiating (1) exteriorly and applying Cartan's lemma, we obtain

$$dl_{iab} - \omega_{ik}l_{kab} - \omega_{ac}l_{icb} - \omega_{bc}l_{iac} = \{R_{ikab} - 1/2(l_{iac}l_{kcb} - l_{ibc}l_{kca})\}\omega_k + 2R_{icab}\omega_c, \quad (2)$$

$$R_{iakj} = 0, \quad R_{iakb} = 1/2R_{ikab} + 1/4(l_{iac}l_{kcb} + l_{ibc}l_{kca}).$$

2. Obviously,

$$D\omega_a = [\omega_b \theta_{ba}], \quad \theta_{ba} = \omega_{ba} + l_{iba} \omega_i.$$

Since  $\theta_{ab} + \theta_{ba} = 0$ , the  $\theta_{ab}$  are forms of a Riemannian connection in the space of leaves  $V^n$ , which we shall call the base. To every curve in the base  $V^n$  there corresponds a one-parameter family of leaves in  $V^{n+m}$ . Since the leaves are totally geodesic submanifolds, the orthogonal trajectories in a one-parameter family of leaves establish an isometric correspondence between the leaves. In particular, to a closed contour in the base there corresponds an isometric mapping of a leaf onto itself. Thus, each leaf, generally speaking, admits a group of motions generated by all closed contours in the base that pass through the point over which the leaf lies. Obviously, the groups of motions of different leaves are isomorphic. This abstract group, acting in each leaf as a group of motions, we shall call the **holonomy group of the Riemannian fibered space**  $V^{n+m}$ .

In this note a description will be given of Riemannian fibered spaces with a one-parameter holonomy group.

3. Thus, let the holonomy group be one-parameter; then in each fiber there is induced a Killing vector field whose flow lines are the trajectories of the group. Moreover, to each trajectory of the fiber over some point of the base there corresponds a unique trajectory of the fiber over any other point of the base. This correspondence is established by means of a path joining two points of the base (over which the fibers lie) and the isometric mapping of the fibers corresponding to this path. Since the holonomy group is one-parameter, under this mapping the correspondence between trajectories is one-to-one (independent of the path joining the two points). Consequently, in  $V^{n+m}$  there is a "transverse" foliation into  $(n+1)$ -dimensional surfaces, each of which intersects a fiber along a trajectory of the holonomy group.

Choosing at each point of a fiber the vector  $\bar{e}_m$  tangent to the trajectory passing through this point, we obtain that the system  $\omega_\alpha = 0$ ,  $\alpha = 1, \dots, m-1$ , is completely integrable. The Killing field in each fiber has the form  $\lambda_\alpha = 0$ ,  $\lambda_m = \xi \neq 0$ . Since under the isometric mapping of a fiber to a fiber along a path in the base the length of a vector is preserved,  $\xi$  must be constant along any transverse surface, i.e.

$$d\xi = \xi_\alpha \omega_\alpha.$$

In consequence of the complete integrability of the system  $\omega_\alpha = 0$  and (2) we have

$$\begin{aligned} \omega_{am} &= a_{\alpha\beta} \omega_\beta + b_a \omega_m, & l_{aab} &= 0, \\ dl_{mab} - \omega_{ac} l_{mcb} - \omega_{bc} l_{mac} &= b_{al_{mab}} \omega_\alpha + 2R_{mcab} \omega_c, & (3) \\ a_{\alpha\beta} + a_{\beta\alpha} &= 0, & R_{\alpha\beta ab} &= -a_{\alpha\beta} l_{mab}, & R_{\alpha mab} &= -b_\alpha l_{mab}, & R_{\alpha cab} &= 0. \end{aligned}$$

The condition that  $\lambda_\alpha = 0$ ,  $\lambda_m = \xi$  is a Killing vector in the fiber gives

$$\xi_\alpha = \xi b_\alpha.$$

Since  $D\omega_\alpha = [\omega_\beta \theta_{\beta\alpha}]$ ,  $\theta_{\beta\alpha} = \omega_{\beta\alpha} - a_{\beta\alpha} \omega_m$  and  $\theta_{\beta\alpha} + \theta_{\alpha\beta} = 0$ , a Riemannian connection with forms  $\theta_{\alpha\beta}$  is defined in the space of the “transverse” fibers  $V^{m-1}$ .

By virtue of relations (3) we obtain:

1°.  $a_{\alpha\beta}$ ,  $b_\alpha$  are tensors in  $V^{m-1}$ , and

$$\nabla b_\alpha = b_{\alpha\beta} \omega_\beta, \quad b_{\alpha\beta} = 2R_{\alpha m \beta m} - a_{\alpha\gamma} a_{\gamma\beta} - b_\alpha b_\beta,$$

$$\nabla a_{\alpha\beta} = a_{\alpha\beta\gamma} \omega_\gamma, \quad a_{\alpha\beta\gamma} = 2R_{\gamma m \alpha\beta} - (b_\alpha a_{\gamma\beta} + b_\beta a_{\alpha\gamma} - b_\gamma a_{\beta\alpha})$$

( $\nabla$  denotes the covariant differentiation in the connection  $\theta_{\alpha\beta}$ );  $\xi$  is a scalar in  $V^{m-1}$ ,

$$d\xi = \xi b_\alpha \omega_\alpha.$$

2°.  $c_{ab} = \frac{l_{mab}}{\xi}$  is a tensor in the base  $V^n$ , and

$$\nabla c_{ab} = c_{abc} \omega_c, \quad c_{abc} = \frac{2R_{mcab}}{\xi}.$$

Obviously,  $b_{[\alpha\beta]} = 0$ ,  $a_{(\alpha\beta\gamma)} = b_{(\alpha} a_{\beta\gamma)}$ ,  $c_{(abc)} = 0$ .

Thus the following has been proved:

**Theorem 1.** If  $V^{n+m}$  is a Riemannian fibered space with a one-parameter holonomy group, then

1) in the base  $V^n$  there is a skew-symmetric tensor  $c_{ab}$  satisfying the condition

$$\nabla c_{ab} = c_{abc} \omega_c, \quad c_{(abc)} = 0; \quad (4)$$

2) in the space of “transversal” layers  $V^{m-1}$  there are a vector  $b_\alpha$ , a skew-symmetric tensor  $a_{\alpha\beta}$ , and a scalar  $\xi \neq 0$ , satisfying the conditions

$$\nabla b_\alpha = b_{\alpha\beta} \omega_\beta, \quad b_{[\alpha\beta]} = 0,$$

$$\nabla a_{\alpha\beta} = a_{\alpha\beta\gamma} \omega_\gamma, \quad a_{(\alpha\beta\gamma)} = b_{(\alpha} a_{\beta\gamma)}, \quad (5)$$

$$d \ln \xi = b_\alpha \omega_\alpha.$$

4. Let us prove the following converse theorem:

**Theorem 2.** Let there be given: 1) a Riemannian space  $V^n$  and in it a skew-symmetric tensor  $c_{ab}$  satisfying condition (4); 2) a Riemannian space  $V^{m-1}$  and in it a vector  $b_\alpha$ , a skew-symmetric tensor  $a_{\alpha\beta}$ , and a scalar  $\xi \neq 0$ , satisfying conditions (5). Then there exists a unique (up to a change of coordinates) fibred Riemannian space  $V^{n+m}$  with a one-parameter holonomy group, for which  $V^n$  is the base and  $V^{m-1}$  the space of “transversal” layers.

For the proof of the theorem it is enough to show that:

1°. There exists a form  $\omega_m$ , independent of  $\omega_a$  and  $\omega_\alpha$ , and satisfying the equation

$$D\omega_m = a_{\alpha\beta}[\omega_\alpha\omega_\beta] + b_a[\omega_a\omega_m] - \xi c_{ab}[\omega_a\omega_b]. \quad (6)$$

2°. Different solutions of (6) determine Riemannian spaces differing only by a change of coordinates.

Equation (6) is closed with respect to exterior differentiation by virtue of (4) and (5). We shall seek a solution of (6) in the form

$$\omega_m = dx_m + A_\alpha\omega_\alpha + A_a\omega_a, \quad (7)$$

where  $A_\alpha, A_a$  are unknown functions, and  $x_m$  is a new independent variable.

Substituting (7) into (6), we obtain an equation of the form

$$[\Delta A_\alpha\omega_\alpha] + [\Delta A_a\omega_a] = 0, \quad (8)$$

where  $\Delta A_\alpha$  and  $\Delta A_a$  are suitable forms. Since equation (6) is closed, (8) is also closed; therefore, for the construction of a regular chain of integral elements we have only one equation (8). Obviously, for the characters of equation (8) we have  $s_1 = s_2 = \dots = s_{n+m-1} = 1$ . The Cartan number

$$Q = 1 + 2 + \dots + n + m - 1 = \frac{(n+m)(n+m-1)}{2}.$$

The genus of the most general integral element is

$$N = \frac{(n+m)(n+m-1)}{2}.$$

$Q = N$ , and equation (8) is in involution [2]. Thus, equation (6) has a solution. It is obvious that a form of the type (7) is linearly independent of  $\omega_a, \omega_\alpha$ .

Let  $\omega_m$  and  $\bar{\omega}_m$  be two solutions of equation (6); then the systems of forms  $(\omega_m, \omega_a, \omega_\alpha)$  and  $(\bar{\omega}_m, \omega_a, \omega_\alpha)$  determine  $V^{n+m}$  and  $\bar{V}^{n+m}$ , respectively. Consider the system of equations  $\omega_m = \bar{\omega}_m, \omega_a = \omega_a, \omega_\alpha = \omega_\alpha$ . By virtue of (6)

it is completely integrable. Consequently, there exists a change of coordinates that carries  $V^{n+m}$  into  $\bar{V}^{n+m}$ . The theorem is proved.

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### CITED LITERATURE

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2. S. P. Finikov, *Cartan's Method of Exterior Forms*, Moscow, 1948, p. 247.

*Note: Figure translations are in progress. See original paper for figures.*

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