ON A SYSTEM OF EQUATIONS IN THE PROBLEM OF BENDING OF A CIRCULAR PRISMATIC BAR WITH AN ECCENTRIC ELLIPTICAL HOLE

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Abstract

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ON A SYSTEM OF EQUATIONS IN THE PROBLEM OF BENDING OF A CIRCULAR PRISMATIC BAR WITH AN ECCENTRIC ELLIPTICAL HOLE

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In paper (^{1}), the solution of the problem of transverse bending of a circular prismatic bar with an eccentric elliptical hole was reduced directly to an infinite system of linear algebraic equations. The latter was effectively solved for relatively close boundaries of the cross section of the bar. This system is studied here, and it is established that it is quasiregular for arbitrary relative dimensions of the cross section and completely regular for boundaries of the section of the bar that are very close to each other and for a sufficiently elongated ellipse (the ratio of the semiaxes is equal to 1/5).

1. Substitute into condition (2) of paper (^{1}), instead of $\varphi(t)$, its expression from (3), taking into account the form of the mapping function (4); then on the circle γ of radius $\rho > 1$ in the ζ-plane we obtain a certain equality. From it we obtain a function holomorphic outside γ, including the point at infinity,

$$\varphi^*(\zeta) = \frac{1}{2\pi i}\int_{\gamma_1} \omega^*(\sigma)\left[\frac{d\sigma}{\sigma^2\left(\zeta+\frac{1}{\sigma}\right)} - \frac{\rho^2 d\bar\sigma}{\bar\sigma^2\left(\zeta-\frac{\rho^2}{\sigma}\right)}\right] - p_3\frac{\rho^3}{\zeta^3} + p_2\frac{\rho^2}{\zeta^2} - p_1\frac{\rho}{\zeta}, \quad (1)$$

where γ_1 is a curve in the ζ-plane that is the image of the circle L_1; it is traversed counterclockwise. The meaning of the remaining notation is given in paper (^{1}).

Multiplying both sides of equality (1) by $(2\pi i)^{-1}Rt^{-(m-1)}\,dt$, assuming the point ζ to lie on γ_1, and integrating over the circle L_1, after some transformations we obtain

$$\bar\beta_m = -\frac{1}{2\pi i R^m}\int_{L_1} \varphi(t)\, t^{m-1}\, dt. \quad (2)$$

To find the sum of the moduli of the coefficients q_{km} of system (17), obtained in paper (^{1}), in any equation of the system we consider the function of the parameter x

$$r_m(x) = \sum_{k=1}^{\infty} |q_{km}| x^k. \tag{3}$$

We obtain it by putting, in the sum contained in (17),

$$\bar{\beta}_k = (-i)^m i^k x^k; \qquad \beta_k = (-i)^m (-ix)^k,$$

since, on the basis of formulas (14) and (17) from (^{1}), we have

$$q'_{km} = (-i)^{m+k} |q_{km}|; \qquad q_{km} = (-1)^k (-i)^{k+m} |q_{km}|.$$

Replacing in the Fourier expansion for $\omega(t)$, respectively, the coefficients $\bar{\beta}_k$ by $(-i)^m (ix)^k$ and β_k by $(-i)^m (-ix)^k$, we shall have

$$\omega(t, x) = \beta_0 - (-i)^m \frac{ixt}{R + ixt} - (-i)^m \frac{ixR}{t - ixR}.$$

Substituting this expression for the function $\omega(t, \chi)$ into expression (1) and discarding the free terms in it, we shall have:

$$\varphi^*(\xi, \chi) = I_1(\xi, \chi) + I_2(\xi, \chi). \tag{4}$$

Here

$$I_1(\xi, \chi) = -\frac{(-i)^m}{2\pi i} \int_{\gamma_1} \left(\frac{i\chi t}{R + i\chi t} + \frac{i\chi R}{t - i\chi R} \right) \frac{d\sigma}{\sigma^2 (\xi + 1/\sigma)};$$

$$I_2(\xi, \chi) = \frac{(-i)^m}{2\pi i} \int_{\gamma_1} \left(\frac{i\chi t}{R + i\chi t} + \frac{i\chi R}{t - i\chi R} \right) \frac{\rho^2 d\bar{\sigma}}{\bar{\sigma}^2 (\xi - \rho^2/\bar{\sigma})}.$$

In $I_1(\xi, \chi)$, the first term of the integrand is equal to

$$\frac{i\chi t}{R + i\chi t} \, \frac{1}{\sigma(\sigma\xi + 1)}. \tag{5}$$

From the form of the mapping function (4) from (1), we find that

$$\frac{i\chi t}{R + i\chi t} = -1 + \frac{R\sigma}{A\chi i(\sigma - \sigma_1(\chi))(\sigma - \sigma_2(\chi))},$$

where

$$\sigma_{1,2}(\chi) = \frac{R - \chi e \mp \sqrt{(R-\chi e)^2 - 4A^2\chi^2}}{2\chi A}\, i.$$

The point $\sigma_1(\chi)$ lies outside γ_1, while the point $\sigma_2(\chi)$ lies inside γ_1, which maps the ellipse.

Thus, expression (5) has simple poles $\sigma = \sigma_1(\chi)$, $\sigma = \sigma_2(\chi)$, $\sigma = 0$, and $\sigma = -1/\xi$; of these, the pole $\sigma = \sigma_1(\chi)$ lies outside γ_1, while the remaining poles lie inside γ_1.

The second term of the same integrand is as follows:

$$\frac{i\chi R}{t - i\chi R}\,\frac{1}{\sigma(\sigma\xi+1)}. \tag{6}$$

From the form (4) from (1), we find that

$$\frac{i\chi R}{t - i\chi R} = \frac{i\chi R\sigma}{A(\sigma - \sigma_3(\chi))(\sigma - \sigma_4(\chi))},$$

where

$$\sigma_{3,4}(\chi) = \frac{-(e-\chi R) \mp \sqrt{(e-\chi R)^2 - 4A^2}}{2A}\, i.$$

Both points $\sigma_3(\chi)$ and $\sigma_4(\chi)$ lie inside γ_1. Thus, expression (5) has simple poles $\sigma = \sigma_3(\chi)$, $\sigma = \sigma_4(\chi)$, $\sigma = 0$, and $\sigma = -1/\xi$, which lie inside the curve γ_1.

Without changing the values of the integrals whose integrands contain (5) and (6), in the first of them one may take as the contour of integration the curve γ_1', lying entirely inside the curve γ_1 and sufficiently close to it (since the pole σ_1 is located outside γ_1, while the remaining ones: σ_2, $\sigma = 0$, and $\sigma = -1/\xi$, lie inside γ_1); in the second integral, as the contour of integration one may take the curve γ_1'', enclosing γ_1, since all poles of the integrand lie inside γ_1.

2. We now proceed to the computation of the new integrals thus obtained, first passing in them to the limit as $\chi \to 1$ (which is possible, since on the curves γ_1' and γ_1'' the modulus $|t| \neq R$). Taking into account that the second of the integrals is equal to zero (since all poles of its integrand lie inside γ_1''), we shall have:

$$I_1(\xi, 1) = (-i)^m \frac{R}{\sqrt{(R-e)^2 - 4A^2}}\,\frac{1}{\sigma_1\xi + 1},$$

where

$$\sigma_{1,2} = \frac{R - e \mp \sqrt{(R-e)^2 - 4A^2}}{2A}\, i.$$

Similarly, we find

$$I_2(\xi, 1) = (-i)^m \frac{R}{\sqrt{(R-e)^2 - 4A^2}} \, \frac{\rho^2}{\sigma_1 \xi + \rho^2}.$$

Substituting the values of $I_1(\xi, 1)$ and $I_2(\xi, 1)$ into (4), we obtain

$$\varphi^*(\xi, 1) = (-i)^m \frac{R}{\sqrt{(R-e)^2 - 4A^2}} \left(\frac{1}{\sigma_1 \xi + 1} + \frac{\rho^2}{\sigma_1 \xi + \rho^2} \right). \tag{7}$$

Next we multiply both sides of (7) by $(2\pi i)^{-1} t^{m-1} R^{-m}\, dt$ and integrate over the circumference L_1. Then we obtain an expression for the integral occurring in (2), in which the function $\varphi(t)$ is replaced by $\varphi^*(\xi, 1)$; denoting this expression by $r_m(1)$, we shall have:

$$r_m(1) = -(-i)^m \frac{R}{\sqrt{(R-e)^2 - 4A^2}} \, \frac{1}{R^m} \, \frac{1}{2\pi i} \int_{L_1} \left(\frac{1}{\sigma_1 \sigma + 1} + \frac{\rho^2}{\sigma_1 \sigma + \rho^2} \right) t^{m-1}\, dt.$$

Passing, by means of the mapping function (4) from (1), to the variable of integration σ instead of t, and taking into account the binomial expansion formulas, we finally arrive at the relation

$$r_m(1) = \frac{R}{\sqrt{(R-e)^2 - 4A^2}} \left(\frac{e}{R} \right)^{m-1} \sum_{\nu=0}^{m-1} C_{m-1}^{\nu} \left(\frac{A}{e} \right)^{\nu} H_\nu(\rho, \sigma_*), \tag{8}$$

where the notation has been introduced

$$H_\nu(\rho, \sigma_*) = \sum_{\nu_1=0}^{\mathrm{E}(\nu/2)} C_\nu^{\nu_1} T_{\nu - 2\nu_1 + 1}(\rho, \sigma_*) - \sum_{\nu_1=0}^{\mathrm{E}(\nu/2)-1} C_\nu^{\nu_1} T_{\nu - 2\nu_1 - 1}(\rho, \sigma_*),$$

with

$$T_n(\rho, \sigma) = \frac{1}{\sigma_*^n} + \left(\frac{\rho^2}{\sigma_*} \right)^n.$$

Here $\sigma_* = -\sigma_1 i$, and $\mathrm{E}(\nu/2)$ is the greatest integer contained in $\nu/2$.

From (14) of (1) we have

$$I_{mn} < \frac{A}{R}\left(\frac{e+2A}{R}\right)^{m-1}.$$

The index n does not occur in the right-hand side of the last inequality; therefore this inequality is valid for any n.

On the basis of (17) of (1) we obtain

$$|q_{km} + q'_{km}| < \frac{A}{R}\left(\frac{e+2A}{R}\right)^{m-1}\sum_{n=1}^{m}\left(|\mu_{kn}| + |\mu^*_{kn}|\right).$$

On the other hand,

$$|\mu_{kn}| + |\mu^*_{kn}| < \left(\frac{e}{R}\right)^{k}(1+\rho^{2n})\sum_{k_1=0}^{k}\left(\frac{2A}{e}\right)^{k_1} C_k^{k_1} < (1+\rho^{2n})\left(\frac{e+2A}{R}\right)^{k}.$$

Thus,

$$|q_{km} + q'_{km}| < \frac{A}{R}\left(\frac{e+2A}{R}\right)^{m+k+1}\sum_{n=1}^{m}(1+\rho^{2n});$$

this proves the regularity of passage to the limit as $\chi \to 1$, i.e.

$$\lim_{\chi\to 1} r_m(\chi) \to r_m(1),$$

where $r_m(1)$ is determined by formula (8).

The estimate for the quantity $r_m(1)$ from formula (8) will be

$$r_m(1) \leq \frac{1+\rho^2}{\sqrt{\left(\frac{R-e}{A}\right)^2 - 1}}\left(\frac{e+a}{R}\right)^{m-1} = r^*_m(1)$$

Hence we conclude that system (17) from (1), since $\dfrac{e+a}{R} < 1$, is always quasiregular.

The sum of the moduli of the coefficients of system (17) from (1), if it is reduced to normal form, will be equal to:

$$r^{**}_m(1) = \frac{r_m(1) - |A_{mm}|}{1 - |A_{mm}|}.$$

The quantities $r_m(1)$, $r^*_m(1)$, $r^{**}_m(1)$ were calculated and tabulated for the relative dimensions of the section $\rho = \sqrt{1.5}$, $e/R = 0.5$, and $A/R = {}^2/_{25}\sqrt{6}$:

m	1	2	3	4	5	6	7	8	9	10
$r_m(1)$	0.7631	0.4755	0.3260	0.2366	0.1781	0.1374	0.1080	0.0860	0.0693	0.0563
$r^*_m(1)$	1.5777	1.4199	1.2779	1.1501	1.0351	0.9316	0.8384	0.7546	0.6791	0.6112
$r^{**}_m(1)$	0.7379	0.4463	0.3144	0.2209	0.1668	0.1292	0.1019	0.0816	0.0659	0.0538

From this table follows the complete regularity of the system of linear algebraic equations for very close boundaries of the cross-section of the bar (the live thickness is 0.1 of the radius of the circle) and a sufficiently elongated ellipse (the ratio of the semiaxes is 1/5).

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Note: Figure translations are in progress. See original paper for figures.

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