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1.** Consider the system of nonlinear differential equations:

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Abstract

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THEORY OF ELASTICITY

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ASYMPTOTICS OF THE EQUATIONS OF LARGE DEFLECTION OF A CIRCULAR SYMMETRICALLY LOADED PLATE

(Presented by Academician Yu. N. Rabotnov, 24 II 1961)

1. Consider the system of nonlinear differential equations:

$$Av - \frac{u^2}{2} = 0, \qquad \varepsilon^2 Au + uv + \varphi(\rho) = 0, \qquad A(\ ) \equiv -\rho \frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho(\ ) \quad (1)$$

with boundary conditions of one of the following forms:

$$v\Big|_{\rho=1} = T > 0, \qquad u\Big|_{\rho=1} = 0; \quad (2a)$$

$$\frac{dv}{d\rho} - \frac{\sigma}{\rho}v\Big|_{\rho=1} = 0, \qquad u\Big|_{\rho=1} = 0; \quad (2b)$$

$$\frac{dv}{d\rho} - \frac{\sigma}{\rho}v\Big|_{\rho=1} = 0, \qquad \frac{du}{d\rho} + \frac{\sigma}{\rho}u\Big|_{\rho=1} = 0; \quad (2c)$$

$$v\Big|_{\rho=1} = 0, \qquad u\Big|_{\rho=1} = 0; \quad (2d)$$

$$\frac{v}{\rho}\Big|_{\rho=0} < \infty, \qquad \frac{u}{\rho}\Big|_{\rho=0} < \infty \quad \left(0 < \sigma < \frac{1}{2}\right).$$

Equations (1)–(2) are the equations of large deflections of a circular axially symmetrically loaded plate ([1,2]). Here v corresponds to the radial force, the function $u = dw/d\rho$, where w is the deflection of the plate, and the boundary conditions (2) correspond to different methods of edge fixing. The quantity $\varepsilon^2 = h^2/12(1-\sigma^2)r_1^2$ characterizes the relative thinness of the plate; h is the thickness; r_1 is the outer radius and σ is Poisson' s ratio.

$$\varphi(\rho) = \frac{1}{Eh}\int_0^\rho q(t)\,t\,dt,$$

where $q(\rho)$ is the intensity of the normal load. The existence and uniqueness of the solution of (1)–(2) follow from the results of ([3,4]).

Along with (1)–(2), consider the membrane equations ($\varepsilon = 0$)

$$Av_0 - \frac{u_0^2}{2} = 0, \qquad u_0 v_0 + \varphi(\rho) = 0 \tag{3}$$

with the corresponding boundary conditions:

$$v_0\big|_{\rho=1} = T; \tag{4a}$$

$$\frac{dv_0}{d\rho} - \frac{\sigma}{\rho} v_0\Big|_{\rho=1} = 0; \tag{4b}$$

$$\frac{dv_0}{d\rho} - \frac{\sigma}{\rho} v_0\Big|_{\sigma=1} = 0; \tag{4c}$$

$$v_0\big|_{\rho=1} = 0; \tag{4d}$$

$$\frac{v_0}{\rho}\Big|_{\rho=0} < \infty.$$

We shall study the boundary-value problem (1)–(2) as $\varepsilon \to 0$. Asymptotic representations of the solutions of problem (1)–(2) as $\varepsilon \to 0$ will be constructed, and it will be shown that the solution of problem (1)–(2) as $\varepsilon \to 0$ in any interior-

of similarity, converges uniformly on $[0, 1]$ to the solution of problem (3)–(4), while the behavior of the solution of problem (1)–(2) in a neighborhood of the point $\rho = 1$ has the character of a boundary layer ([5,6]).

For the special case of the conditions (2b), when $q = \text{const}$, formal asymptotic representations of the solution were constructed in ([7,10]), and the question of justification was posed.

2. As an example, let us consider the case of the boundary conditions (2a). The methods for constructing the asymptotics are, in the main, analogous to those developed in ([6]). We introduce notation: let the vector $\mathbf{V} \equiv (v, u)$ be a solution, and let $P[\mathbf{V}]$ be the left-hand side of system (1).

For the solution of (1), asymptotic representations of the form are constructed

$$v = \sum_{s=0}^{n+2} \varepsilon^s v_s + \sum_{s=0}^{n+2} \varepsilon^s h_s + \sum_{s=0}^{n+2} \varepsilon^s \alpha_s + R_n,$$

$$u = \sum_{s=0}^{n} \varepsilon^s u_s + \sum_{s=0}^{n} \varepsilon^s g_s + \sum_{s=0}^{n} \varepsilon^s \beta_s + S_n. \tag{5}$$

The functions $v_s(\rho)$, $u_s(\rho)$ are obtained with the aid of the 1st iteration process*. Namely, we require that $P[\mathbf{V}_n] = O(\varepsilon^{n+1})$ ($\mathbf{V}_n \equiv (v^n, u^n)$,

$$v^n = \sum_{s=0}^{n} \varepsilon^s v_s, \quad u^n = \sum_{s=0}^{n} \varepsilon^s u_s \Bigg).$$

Equating to zero the coefficients of different powers of ε, we obtain, for determining v_0, u_0, (3), (4a), and for v_s, u_s the system

$$Av_s - \frac{1}{2} \sum_{k+j=s} u_k u_j = 0; \qquad \sum_{k+j=s} u_k v_j + Au_{s-2} = 0; \tag{6}$$

$$u_{-2} = u_{-1} = 0; \qquad s = 1, 2, \dots, n,$$

with boundary conditions

$$\left.\frac{v_s'}{\rho}\right|_{\rho=0} < \infty, \qquad v_s|_{\rho=1} = B_s,$$

where B_s are as yet unknown constants. Integration of (6) for each s is reduced to the solution of a linear equation of the form

$$Av_s + \frac{u_0^2}{v_0} v_s = f_s(\rho); \qquad \left.\frac{v_s'}{\rho}\right|_{\rho=0} < \infty; \qquad v_s|_{\rho=1} = B_s, \tag{7}$$

where $f_s(\rho)$ are known if $v_0, u_0;\ v_1, u_1;\ \dots,\ v_{s-1}, u_{s-1}$ have already been found.

Let us first prove the unique solvability of problem (3)–(4a). For this purpose we introduce a new function $w_0 = v_0/\rho - T$. From (3)–(4a) we obtain

$$w_0 = \int_\rho^1 \frac{1}{t^3}\, dt \int_0^t \frac{\varphi^2(\tau)}{\tau\,(w_0 + T)^2}\, d\tau \equiv Lw_0. \tag{8}$$

Let S_R be the sphere in the space $C[0,1]$: $w \in S_R$ if $\|w\|_C \le R$; K the cone of positive functions; $K_R = S_R \cap K$. K_R is a convex set. It is easy to show that the operator L is continuous and maps K_R into its compact part. Then, by Schauder' s theorem ([8]), we obtain that equation (8) has a solution.

It follows from (8) that $v_0 = \rho w_0 \geq 0$. This now makes it possible to prove the uniqueness of the solution of problem (3)–(4a), and also to establish the unique solvability of problem (7). We note that, in the case of problem (3), (4d), for the existence of a solution it is necessary to assume that $\varphi(1) = 0$.

Functions of boundary-layer type $h_s(\rho)$, $g_s(\rho)$ are obtained with the aid of the 2nd iteration process. For this purpose the differences $v - v^n$ and $u - u^n$ are sought

Here, as in what follows, we use the terminology introduced in (5).

in the form

$$v - v^n = \sum_{m=0}^{n} \varepsilon^m h_m, \qquad u - u^n = \sum_{j=0}^{n} \varepsilon^j g_j. \tag{9}$$

Next let $r = 1 - \rho$ and let $v_k = \sum_{l=0}^{n} v_{kl} r^l$, $u_k = \sum_{l=0}^{\infty} u_{kl} r^l$ be the corresponding Taylor expansions at the point $r = 0$.

To determine g_s, h_s we obtain systems of linear differential equations with constant coefficients:

$$\frac{d^2 h_i}{dt^2} = 0, \qquad h_i|_{t=\infty} \quad (i = 0, 1); \tag{10}$$

$$\frac{d^2 h_{s+2}}{dt^2} = R_1 h_{s+1} + R_2 h_s - \sum_{k+j+l=s} t^l u_{kl} g_j + \sum_{k+j+l+1=s} t^{l+1} u_{kl} g_j - \frac{1}{2} \sum_{j+l=s} g_{jg} i + \frac{1}{2} \sum_{j+l+1=s} t g_{jg} i,$$

$$\frac{d^2 g_s}{dt^2} - v_{00} g_s = R_1 g_{s-1} + R_2 g_{s-2} + \sum_{k+j+l=s} t^l v_{kl} g_j - \sum_{k+j+l+1=s} t^{l+1} v_{kl} g_j + \sum_{j+m=s} g_{jh} m - \sum_{j+m+1=s} t g_{jh} m + \sum_{k+m+l=s} t^l u_{kl} h_m - \sum_{k+m+l+1=s} t^{l+1} u_{kl} h_m, \tag{11}$$

where

$$R_1(\,) \equiv 2t \frac{d^2(\,)}{dt^2} + \frac{d(\,)}{dt}, \qquad R_2(\,) \equiv -t^2 \frac{d^2(\,)}{dt^2} - t \frac{d(\,)}{dt} + (\,),$$

$$g_{-2} = g_{-1} = 0, \qquad v_{00} = T > 0 \quad (s = 0, 1, \ldots)$$

with boundary conditions $g_s|_{t=0} = -u_{s0}$; $g_s|_{t=\infty} = 0$; $h_{s+2}|_{t=\infty} = 0$. From (10) it is clear that $h_0 = h_1 = 0$. We now determine the constants B_s, setting $B_s = -h_s(0)$.

By virtue of (10) we obtain from (11) that

$$g_0(\rho) = -u_{00} e^{-\sqrt{T}\,\frac{1-\rho}{\varepsilon}},$$

i.e. g_0 is a boundary-layer function of zero order.

The functions $\alpha_s(\rho)$ and $\beta_s(\rho)$ compensate for the discrepancy in the fulfillment of the boundary conditions at $\rho = 0$, respectively for the functions $h_s(\rho)$ and $g_s(\rho)$. α_s, β_s are infinitely differentiable and satisfy the conditions

$$\alpha_s = \begin{cases} -h_s(0), & \text{if } 0 \le \rho \le 1/10, \\ 0, & \text{if } 1/5 \le \rho \le 1, \end{cases} \qquad \beta_s = \begin{cases} -g_s(0), & \text{if } 0 \le \rho \le 1/10, \\ 0, & \text{if } 1/5 \le \rho \le 1. \end{cases}$$

3. Set $\varphi_k = v - R_k$ and $\psi_k = u - S_k$. In proving convergence we proceed from the estimate

$$A(v - \varphi_k) - \frac{1}{2}(u^2 - \psi_k^2) = O(\rho\varepsilon^{k+1})^*,$$

$$\varepsilon^2 A(u - \psi_k) + (uv - \varphi_k\psi_k) = O(\rho\varepsilon^{k+1}). \tag{12}$$

Lemma 1 (N. F. Morozov [4]). For the solution of problem (1)–(2a), $v \ge 0$.

Lemma 2. For sufficiently small ε $(0 < \varepsilon < \varepsilon_1)$, for all $\rho \in [0, 1]$ the following hold: 1) $\varphi_k \ge 0$; 2) $\min(\varphi_k/\rho) > T/2$.

Lemma 3. For R_k and S_k the following energy estimate is valid:

$$\int_0^1 \left|\frac{dR_k}{d\rho}\right|^2 d\rho + \frac{1}{2}\int_0^1 \frac{R_k^2}{\rho^2}\,d\rho + \varepsilon^2 \int_0^1 \left|\frac{dS_k}{d\rho}\right|^2 d\rho + \frac{\varepsilon^2}{2}\int_0^1 \frac{S_k^2}{\rho^2}\,d\rho + \frac{T}{4}\int_0^1 S_k^2\,d\rho \le$$

$$\le C\varepsilon^{k+1}\int_0^1 (|R_k| + |S_k|)\,d\rho. \tag{13}$$

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* We say that $f(\rho, \varepsilon) = O(\rho\varepsilon^{k+1})$ if $|f(\rho, \varepsilon)| \le C\rho\varepsilon^{k+1}$, where C is a constant independent of ρ and ε.

Theorem 1. For problem (1), (2a), the asymptotic representations (5) hold, where for R_k and S_k the following estimates are valid:

$$\max_{0\le\rho\le1}|R_k(\rho)|\le C_1\varepsilon^{k+1},\qquad \max_{0\le\rho\le1}|S_k(\rho)|\le C_2\varepsilon^{k+1/2}\quad(k=0,1,2,3,\dots,n);\tag{14}$$

$$\max_{0\le\rho\le1}\left|\frac{dR_k}{d\rho}\right|\le C_3\varepsilon^{k+1}\quad(k=0,1,2,\dots);$$

$$\max_{0\le\rho\le1}\left|\frac{dS_k}{d\rho}\right|\le C_4\varepsilon^{k-1}\quad(k=2,3,\dots);\tag{15}$$

$$\max_{0\le\rho\le1}\left|\frac{d^2R_k}{d\rho^2}\right|\le C_5\varepsilon^{k-1/2}\quad(k=1,2,\dots);$$

$$\max_{0\le\rho\le1}\left|\frac{d^2S_k}{d\rho^2}\right|\le C_6\varepsilon^{k-5/2}\quad(k=3,4,\dots).\tag{16}$$

4. Expansions of the form (5) are also valid for the problems (1), (2b); (1), (2c) and (1), (2d). In this case, for (1), (2b),

$$g_0=-u_{00}\exp\left[-\sqrt{v_{00}}\,\frac{1-\rho}{\varepsilon}\right],\qquad v_{00}>0\tag{17}$$

and the estimates (14)–(16) are valid. In the case (1)–(2c),

$$g_0=-\frac{M\varepsilon}{\sigma\varepsilon+\sqrt{v_{00}}}\exp\left[-\sqrt{v_{00}}\,\frac{1-\rho}{\varepsilon}\right],\tag{18}$$

where

$$M=\frac{du_0}{d\rho}+\frac{\sigma}{\rho}u_0\bigg|_{\rho=1}.$$

In the case (1), (2d),

$$\frac{d^2g_0}{dt^2}-v_{01}\varepsilon t g_0=0;\qquad v_{01}>0,\quad g_0|_{t=0}=-u_{00},\quad g_0|_{t=\infty}=0.\tag{19}$$

The solution of equation (19) is expressed in terms of the Airy function [9].

5. The width of the boundary-effect region is $O(\varepsilon|\ln\varepsilon|)$.

6. The constructions indicated and the proofs of convergence extend to equation (1) with the more general boundary conditions (7)

$$\frac{dv}{d\rho} - \left(\sigma - \frac{\varepsilon}{k_1}\right)\frac{v}{\rho}\bigg|_{\rho=1} = 0, \qquad \frac{du}{d\rho} + \left(\sigma + \frac{k_2}{\varepsilon^3}\right)\frac{u}{\rho}\bigg|_{\rho=1} = 0,$$

$$\frac{v}{\rho}\bigg|_{\rho=0} < \infty, \qquad \frac{u}{\rho}\bigg|_{\rho=0} < \infty. \tag{20}$$

For g_0 we obtain

$$g_0 = -\frac{M_1}{\sqrt{v_{00}\varepsilon^2 + \sigma\varepsilon^3 + k_2}} \exp\left[-\sqrt{v_{00}}\,\frac{1-\rho}{\varepsilon}\right], \tag{21}$$

where

$$M_1 = \varepsilon^3 \frac{du_0}{d\rho} + \left(\varepsilon^3\sigma + k_2\right) u_0\bigg|_{\rho=1}.$$

7. The method used also makes it possible to consider the equations of large deflections of annular plates.

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Note: Figure translations are in progress. See original paper for figures.