PROOF OF THE MINIMALITY OF A CONTACT REALIZATION OF ONE CLASS OF BOOLEAN FUNCTIONS OF n VARIABLES

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Abstract

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PROOF OF THE MINIMALITY OF A CONTACT REALIZATION OF ONE CLASS OF BOOLEAN FUNCTIONS OF n VARIABLES

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The minimality of the number of contacts in a contact circuit has been proved only for two Boolean functions of n variables. Namely, Cardo ([1]) showed that contact circuits realizing the functions $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ and $\bar{x}_1 \oplus x_2 \oplus \cdots \oplus x_n$ contain $4n-4$ contacts, and that this number is minimal. Other proofs ([2]) of the minimality of a contact realization apply to circuits realizing Boolean functions of no more than 4 variables.

This note contains a proof of the minimality of a contact realization of one class of Boolean functions for arbitrary n.

1. Consider the Boolean function

$$C_n = \bar{x}_1\bar{x}_2 \dots \bar{x}_n + x_1\bar{x}_2 \dots \bar{x}_n + \cdots$$

$$\cdots + x_1x_2 \dots x_{n-1}\bar{x}_n + x_1x_2 \dots x_n.$$

In ([3]) such a function is called a chain.

Lemma. In the perfect disjunctive normal form (p.d.n.f.) of the function C_n, the variable x_i occurs i times with negation and $n+1-i$ times without negation.

Theorem 1. C_n is a function that is special ([4]) only with respect to the variables x_1 and x_n.

Proof. Let

$$C_n = x_i^{\sigma_i} f_1(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) + f_2(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) =$$

$$= x_i^{\sigma_i} f_1 + x_i^{\sigma_i} f_2 + \bar{x}_i^{\sigma_i} f_2 = x_i^{\sigma_i}(f_1 + f_2) + \bar{x}_i^{\sigma_i} f_2.$$

In the p.d.n.f. of the function C_n there are only two terms adjacent with respect to x_i; therefore in the p.d.n.f. of the function f_2 there is only one term. Since schematic contact circuit

Figure 1: schematic contact circuit

([4]) $f_1 \cdot f_2 = 0$, and in all there are $n+1$ terms in the p.d.n.f. of the function C_n, it follows that the p.d.n.f. of the function $f_1 + f_2$ contains exactly n terms. Consequently, in the p.d.n.f. of the function C_n there are exactly n terms containing $x_i^{\sigma_i}$. If $\sigma_i = 1$, then in the p.d.n.f. of the function C_n there are exactly n terms containing x_i without negation. Hence, by the lemma, $i = 1$. If $\sigma_i = 0$, then in the p.d.n.f. of the function C_n there are exactly n terms containing x_i with negation. Hence, by the lemma, $i = n$.

Theorem 2. No contact circuit realizing the function C_n can contain fewer than two contacts of the variable x_i, if $i \neq 1$ and $i \neq n$.

Proof. If the variable x_i is represented in the circuit by only one contact, then (cf. ([4])) every path between the poles either

passes, or does not pass, through x_i. But then $C_n = x_i^{\sigma_i} f_1 + f_2$, which for $i \neq 1$ and $i \neq n$ contradicts Theorem 1.

Theorem 3*.

$$(x_1 + \bar{x}_2)(x_2 + \bar{x}_3) \cdots (x_{n-1} + \bar{x}_n) = C_n.$$

Proof. For $n = 2$ the theorem is true. Suppose

$$(x_1 + \bar{x}_2)(x_2 + \bar{x}_3) \cdots (x_{n-2} + \bar{x}_{n-1}) = C_{n-1}.$$

Then

$$\begin{aligned}(x_1 + \bar{x}_2)(x_2 + \bar{x}_3) \cdots (x_{n-1} + \bar{x}_n) &= C_{n-1}(x_{n-1} + \bar{x}_n) \\ &= C_{n-1}\bar{x}_n + C_{n-1}x_{n-1} = C_{n-1}\bar{x}_n + x_1 x_2 \cdots x_{n-1} \\ &= C_{n-1}\bar{x}_n + x_1 x_2 \cdots x_n = C_n.\end{aligned}$$

Corollary. The minimum number of contacts in a circuit realizing the function C_n is equal to $2n - 2$. The circuit has the form:

2. In ([5]) it is shown that the function C_n has an inversion group G of second order. Hence it follows that the number of functions of the same type as C_n is $2^{n-1} \cdot n!$, and for all of them the minimum contact realization is equal to $2n - 2$.

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REFERENCES

1. C. Cardot, Ann. Telecommun., **7**, 2, 75 (1952).

2. Yu. L. Vasil' ev, DAN, **127**, No. 2, 242 (1959).

3. S. V. Yablonskii, Matem. sborn., **30** (72), 2, 329 (1952).

4. I. N. Povarov, Sborn. Problemy peredachi informatsii, vol. 4, 1959.

5. Yu. L. Sagalovich, Sborn. Problemy peredachi informatsii, vol. 8 (1961).

* This representation of the function C_n was proposed by V. N. Roginskii.

Note: Figure translations are in progress. See original paper for figures.