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Abstract

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REPRESENTATIONS OF THE GROUP $Z_2 \times Z_2$ OVER A FIELD OF CHARACTERISTIC 2 *

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O. Higman (^{2}) showed that if a Sylow p-subgroup of a finite group G is not cyclic, then the group G has infinitely many inequivalent indecomposable representations by matrices with entries from a field k of characteristic p. However, the problem of a complete description of all representations is still far from being solved. Even in the simplest situation, when $G = Z_2 \times Z_2$ and the field k has characteristic 2, as far as we know, not all representations have been found.

In Section 1 of this paper all indecomposable representations of the group $Z_2 \times Z_2$ over an algebraically closed field k of characteristic 2 are found. Further, the set of equivalence classes of representations of a group G over a field k forms an associative semiring: addition is induced by the direct sum of representations $\oplus$, and multiplication by the tensor product of representations $\otimes$. The minimal ring $\mathfrak{A}$ containing this semiring is called the **representation ring** of the group G over the field k.

In Section 2 the structure of this ring is studied for $G = Z_2 \times Z_2$. In particular, it turns out that this ring has an infinite number of generators.

1. Let G be a p-group, $\mathfrak{D} = k[G]$ the group algebra of the group G over a field k of characteristic p; I the ideal of the algebra $\mathfrak{D}$ consisting of elements of the form $\sum_{\sigma \in G} \alpha_\sigma \sigma$, where $\sum_{\sigma \in G} \alpha_\sigma = 0$. The $\mathfrak{D}$-module I is a free k-module of rank $p^s - 1$ ($p^s = \operatorname{ord} G$). Let S be the ideal consisting of elements $\gamma \sum_{\sigma \in G} \sigma$, $\gamma \in k$. As a $\mathfrak{D}$-module the ideal S is isomorphic to the $\mathfrak{D}$-module k, on which the elements of G act trivially. The quotient module $J = \mathfrak{D}/S$ is a free k-module of rank $p^s - 1$. Denote by Sp the mapping
$$a \mapsto \sum_{\sigma \in G} \sigma a, \quad a \in A.$$

Lemma 1. *If A is an indecomposable $\mathfrak{D}$-module distinct from $\mathfrak{D}$, then* $\operatorname{Sp}(A) = (0)$.

The proof of this lemma is given in (^{1}).

Remark 1. It is clear that all unitary $\mathfrak{D}$-modules are essentially I-modules, since any I-module can be made into a $\mathfrak{D}$-module by putting $e \cdot a = a$, where e is the identity of the group G and $a \in A$. If $\operatorname{Sp}(A) = (0)$, then the I-module is in fact an N-module, where $N = I/S$. Therefore, by Lemma 1, all indecomposable

$\mathfrak{D}$-modules distinct from $\mathfrak{D}$ are N-modules. In the language of representation theory this means that the problem of finding all indecomposable representations of the group G is equivalent to the problem of finding all representations of the algebra N.

If now $G = Z_2 \times Z_2$, σ, τ are its generators and the field k has characteristic 2, then the ideal I is generated by the elements $\sigma - e$, $\tau - e$, where e is the identity of the group G, and $S = I^2$. Therefore $N = I/I^2$ is an algebra with two generators ν_1 and ν_2 and zero multiplication. Denote by A' the submodule of the module A consisting of all elements $a \in A$ such that $\nu \cdot a = 0$ $(\nu \in N)$, and let $A'' = A/A'$. Then for elements $a'' \in A''$ we have $\nu \cdot a'' = 0$ $(\nu \in N)$.

Lemma 2. *If A and B are two N-isomorphic N-modules, then an N-isomorphism $A \xrightarrow{\varphi} B$ induces an N-isomorphism $A' \xrightarrow{\varphi'} B'$.*

* The results of this work were reported at the seminar on algebraic geometry and homological algebra in Uzhgorod in October 1959.

The proof of Lemma 2 is obvious.

The considerations carried out above in the language of module theory show that, in a suitable basis, every representation of the algebra N has the form

$$v_i \to \begin{pmatrix} 0_{nn} & A_{nm}^{(i)} \\ 0_{mn} & 0_{mm} \end{pmatrix}, \qquad i = 1, 2, \tag{1}$$

where $A_{nm}^{(i)}$ $(i = 1, 2)$ are rectangular (n, m)-matrices having no common annihilating vectors; $0_{nn}, 0_{mn}, 0_{mm}$ are zero matrices; moreover, by Lemma 2, different bases in which the representation has the form (1) are connected by a linear transformation with matrix

$$Q = \begin{pmatrix} P_{nn} & * \\ 0_{mn} & P_{mm} \end{pmatrix},$$

where P_{nn}, P_{mm} are nonsingular square matrices, and 0_{mn} is the zero matrix. Therefore equivalent representations of type (1) have the form

$$v_i \to \begin{pmatrix} 0_{nn} & P_{nn}^{-1} A_{nm}^{(i)} P_{mm} \\ 0_{nm} & 0_{nn} \end{pmatrix}, \qquad i = 1, 2.$$

Thus, every representation of the algebra N is determined by a pair of rectangular matrices $A_{nm}^{(i)}$ $(i = 1, 2)$ having no common annihilating vector, and equivalent representations are determined by a pair of matrices $A_{nm}^{\prime(i)}$ $(i = 1, 2)$ such that $A_{nm}^{\prime(i)} = P_{nn}^{-1} A_{nm}^{(i)} P_{mm}$. The pair of matrices $A_{nm}^{\prime(i)}$ $(i = 1, 2)$ is called **strictly equivalent** to the pair of matrices $A_{nm}^{(i)}$ $(i = 1, 2)$.

Thus, the problem of finding all representations of the algebra N is reduced to the problem of reducing a pair of matrices $A_{nm}^{(i)}$ $(i = 1, 2)$, or, what is the same thing, the pencil of matrices $A_{nm} = A_{nm}^{(2)} + \lambda A_{nm}^{(1)}$, to a strictly equivalent canonical form. Since the pair of matrices $A_{nm}^{(i)}$ $(i = 1, 2)$ has no common annihilating vector, there is no linear dependence with constant coefficients among the rows of the pencil A_{nm}. Such a pencil (3) is strictly equivalent to a pencil of the form:

$$(0_{ng}, A_{n,m-g}^{(2)}) + \lambda(0_{ng}, A_{n,m-g}^{(1)}), \tag{2}$$

where the pencil $A_{n,m-g} = A_{n,m-g}^{(2)} + \lambda A_{n,m-g}^{(1)}$ has the quasidiagonal form

$$A_{n,m-g} = \{L_{\varepsilon_1}, \dots, L_{\varepsilon_r}; L'_{\eta_1}, \dots, L'_{\eta_s}; N^{(1)}, \dots, N^{(t)}, J_1 + \lambda E_1, \dots, \dots, J_u + \lambda E_u\}.$$

Here

$$L_{\varepsilon_i} = \left(\begin{array}{ccccccc} \lambda & 1 & 0 & \cdots & \cdots & 0 & \\ 0 & \lambda & 1 & 0 & \cdots & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & \cdot & \cdot & \cdot & \lambda & 1 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & \lambda & 1 \end{array}\right\} \varepsilon_i,$$

$$L'_{\eta_i} = \left(\begin{array}{cccccc} \lambda & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \lambda & 0 & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \lambda & 0 \\ 0 & \cdot & \cdot & 1 & \lambda & \\ 0 & 0 & \cdot & \cdot & 0 & 1 \end{array}\right\} \eta_i + 1,$$

$$N^{(i)} = \left(\begin{array}{cccccc} 1 & \lambda & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \lambda & 0 \\ 0 & \cdot & \cdot & 1 & \lambda & \\ 0 & 0 & \cdot & \cdot & 0 & 1 \end{array}\right\} \nu_i,$$

$$J_i = \left(\begin{array}{cccccc} \lambda_i & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \lambda_i & 1 & \\ 0 & 0 & \cdot & \cdot & 0 & \lambda_i \end{array}\right\} \mu_i.$$

At the same time it is assumed that the entries of the matrices of the pencil A_{nm} lie in the algebraically closed field k.

We shall call a pencil A, among whose rows there is no linear dependence with constant coefficients, **decomposable** if it is strictly equivalent to a pencil of the form (2) or to a pencil of quasidiagonal form $\{A_1, A_2\}$. The representation (1) of the algebra N corresponding to the pencil A will be denoted by $d(A)$.

Lemma 3. The representation $d(A)$ is decomposable if and only if the corresponding pencil A is decomposable.

The proof of this lemma presents no difficulties.

It remains to note that from the uniqueness of the canonical form of a pencil and from Lemma 3 it follows that the representations $d(L_{\varepsilon_i})$, $d(L'_{\eta_i})$, $d(N^{(i)})$, $d(J_i+\lambda E_i)$ of the algebra N are indecomposable and that, together with the zero representation, they exhaust all indecomposable representations of the algebra N. Therefore, taking Remark 1 into account, the following may be regarded as proved:

Proposition 1. All indecomposable nonequivalent representations of the group $Z_2 \times Z_2$ over an algebraically closed field k of characteristic 2 are exhausted by the following representations:

1) The unit representation

$$\varepsilon(\sigma) = 1, \qquad \varepsilon(\tau) = 1.$$

2) Representations of order $2s+1$, $s = 1, 2, \dots,$

$$I_{2s+1}(\sigma) = \begin{pmatrix} E_{s+1} & L'_s(0) + L'_s(1) \\ & E_s \end{pmatrix}, \qquad I_{2s+1}(\tau) = \begin{pmatrix} E_{s+1} & L'_s(0) \\ & E_s \end{pmatrix}.$$

$$J_{2s+1}(\sigma) = \begin{pmatrix} E_s & L_s(0) + L_s(1) \\ & E_{s+1} \end{pmatrix}, \qquad J_{2s+1}(\tau) = \begin{pmatrix} E_s & L_s(0) \\ & E_{s+1} \end{pmatrix}.$$

3) Representations of order $2s$, $s = 1, 2, \dots,$

$$\Delta_{2s}(\lambda)(\sigma) = \begin{pmatrix} E_s & E_s \\ & E_s \end{pmatrix}, \qquad \Delta_{2s}(\lambda)(\tau) = \begin{pmatrix} E_s & J_s(\lambda) \\ & E_s \end{pmatrix}, \qquad \lambda \in k, \quad s = 1, 2, \dots,$$

$$\Delta_{2s}(\infty)(\sigma) = \begin{pmatrix} E_s & J_s(0) \\ & E_s \end{pmatrix}, \qquad \Delta_{2s}(\infty)(\tau) = \begin{pmatrix} E_s & E_s \\ & E_s \end{pmatrix}.$$

4) The regular representation R of order 4.

2. As before, $G = Z_2 \times Z_2$; k is a field of characteristic 2. The k-module I in this case is a free k-module of rank 3. The representation of dimension 3 of the group G corresponding to this module will also be denoted by I. J is also a free k-module of rank 3. The corresponding representation of dimension 3 will likewise be denoted by J.

Remark 2. One can verify that the representation I is equivalent to the representation I_3, and the representation J is equivalent to the representation J_3, obtained by us in Proposition 1. In [1] it is proved that the representation I is dual to the representation J. It turns out that the representation I_{2s+1} is also dual to the representation J_{2s+1}. Obviously, R is dual to itself.

In the subsequent computations the following will be useful.

Lemma 4. Let G be a p-group, and let k be a field of characteristic p. Then the regular representation R of the group G is contained as a direct summand in a representation D as many times as the rank of the matrix

$$\operatorname{Sp} D = \sum_{\sigma \in G} D(\sigma).$$

This lemma is easily proved with the aid of Lemma 1.

Let us now proceed to the computation of the representation ring $\mathfrak{A}$ of the group G. Obviously, ε is the identity of this ring. Moreover, it is known that

$$D \otimes R = \underbrace{R \oplus \cdots \oplus R}_{m \text{ times}} = mR,$$

where m is the dimension of the representation D.

Proposition 2.

$$I_{2s+1} \otimes I_{2t+1} = I_{2(s+t)+1} \oplus stR,$$

$$J_{2s+1} \otimes J_{2t+1} = J_{2(s+t)+1} \oplus stR,$$

$$I_{2s+1} \otimes J_{2t+1} = I_{2(s-t)+1} \oplus (st+t)R, \quad \text{if } s > t.$$

$$I_{2s+1} \otimes J_{2s+1} = \varepsilon \oplus (s^2+s)R,$$

$$I_{2s+1} \otimes J_{2t+1} = J_{2(t-s)+1} \oplus (st+s)R, \quad \text{if } s < t.$$

The **proof** is based on Lemma 4, Proposition 1, and the results of [1].

Lemma 5.

$$\Delta_{2n}(\lambda) \otimes \Delta_{2m}(\mu) = nmR, \qquad \text{if } \lambda \neq \mu,$$

$$\Delta_{2n}(\lambda) \otimes \Delta_{2m}(\lambda) = (nm - \min(m,n))R \oplus D_{4\min(m,n)},$$

where $D_{4\min(m,n)}$ is a representation of the group G of dimension $4\min(m,n)$, not containing R.

This lemma is easily proved with the aid of Lemma 4.

Proposition 3.

$$I_{2s+1} \otimes \Delta_{2t}(\lambda) = \Delta_{2t}(\lambda) \oplus stR,$$

$$J_{2s+1} \otimes \Delta_{2t}(\lambda) = \Delta_{2t}(\lambda) \oplus stR.$$

The **proof** is based on Lemmas 4, 5, Propositions 1, 2, and the results of [1].

The following lemma, which we shall state without proof, will be needed later.

Lemma 6. Let $\mathfrak{A}$ be an associative, commutative ring with generators x_s ($s = 1, 2, \ldots;\ x_0 = 0$), on which an integer-valued linear functional f is defined such that $f(x_s) = s$, $f(x_s \cdot x_t) = 2\min(s,t)$. Then $x_m \cdot x_n = 2x_{\min(m,n)}$, if $m \neq n$.

Proposition 4.

$$\Delta_{2n}(\lambda) \otimes \Delta_{2m}(\mu) = nmR, \qquad \lambda \neq \mu. \tag{3}$$

For $\lambda \neq 0, 1, \infty$:

$$\Delta_{2n}(\lambda) \otimes \Delta_{2m}(\lambda) = (nm - \min(m,n))R \oplus 2\Delta_{2\min(m,n)}(\lambda), \quad m \neq n; \tag{4}$$

$$\Delta_{2n}(\lambda) \otimes \Delta_{2n}(\lambda) = \begin{cases} (n^2 - n)R \oplus 2\Delta_{2n}(\lambda), & n \equiv 0\ (2), \\ (n^2 - n)R \oplus \Delta_{2(n-1)} \oplus \Delta_{2(n+1)}, & n \equiv 1\ (2); \end{cases} \tag{5}$$

For $\lambda = 0, 1, \infty$:

$$\Delta_{2n}(\lambda) \otimes \Delta_{2m}(\lambda) = (nm - \min(m,n))R \oplus 2\Delta_{2\min(m,n)}(\lambda). \tag{6}$$

Proof. Formula (3) was established by us in Lemma 5. It was also established there that R is contained in $\Delta_{2n}(\lambda) \otimes \Delta_{2m}(\lambda)$ exactly $(nm - \min(n,m))$ times. Let $\mathfrak{A}$ be the representation ring of the group G, W the ideal generated by the regular representation R, and $\mathfrak{A}^* = \mathfrak{A}/W$. Using the results of Proposition 3 and Lemma 5, we find that, for fixed λ, the linear subspace $\mathfrak{B}^*_\lambda$, generated by the classes $x_s = \{\Delta_{2s}(\lambda) \bmod W\}$, forms a subring of the ring $\mathfrak{A}^*$. Define on this ring an integer-valued linear functional f, putting $f(x_s) = s$. According to Lemma 5, $f(x_s \cdot x_t) = 2\min(s,t)$, $s \neq t$. To obtain (4), it remains to apply Lemma 6 to the ring $\mathfrak{B}^*_\lambda$. For the proof of relations (5) and (6) it is necessary to carry out rather lengthy computations, essentially using the structure of the representations $\Delta_{2s}(\lambda)$. We omit these computations. This completes the proof of Proposition 4.

It remains to say that Propositions 2, 3, and 4 completely determine the structure of the ring $\mathfrak{A}$. In particular, it is clear that this ring has no finite number of generators.

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Note: Figure translations are in progress. See original paper for figures.