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Abstract

Full Text

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ON THE MULTIPLICATIVE REPRESENTATION OF SOME ANALYTIC OPERATOR FUNCTIONS

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1. In the works of V. P. Potapov ([1,2]) the problem of the multiplicative representation of analytic and J-nonexpanding matrix functions in the upper half-plane was solved. Subsequently, M. S. Livshits ([3]) and Yu. P. Ginzburg ([4]) investigated certain generalizations of this problem to the infinite-dimensional case.

Consider an operator function $W(\lambda)$ possessing the following properties:

I. The function $W(\lambda)$ is holomorphic in the domain G, obtained by removing from the extended complex plane a certain bounded set of real points.

II. In a neighborhood of the infinitely distant point the function $W(\lambda)$ is expanded in a norm-convergent series

$$W(\lambda) = E + \frac{1}{\lambda} W_1 + \frac{1}{\lambda^2} W_2 + \cdots,$$

where W_k $(k = 1, 2, \ldots)$ are completely continuous operators acting in the separable Hilbert space $\mathfrak{H}_W$.

III. There exists an operator J such that $J = J^*$; $J^2 = E$; $W^*(\lambda) J W(\lambda) - J \geqslant 0,\ \operatorname{Im}\lambda > 0$; $W^*(\lambda) J W(\lambda) - J = 0,\ \operatorname{Im}\lambda = 0,\ \lambda \in G$.

Let us note that from II and III it follows that the operator T, defined by the equality $W_1 = 2iTJ$, is positive.

The theorems of M. S. Livshits and Yu. P. Ginzburg make it possible to represent the operator function $W(\lambda)$ multiplicatively only in the case when the trace of the operator T converges. More general assertions, formulated below, are obtained by means of methods from the theory of non-self-adjoint operators.

Theorem 1. *If the operator function $W(\lambda)$ possesses properties I, II, III and satisfies the condition* *

$$\sum_n \frac{|\omega_n|}{n} < \infty, \tag{1}$$

where ω_n $(n = 1, 2, ...)$ are the nonzero eigenvalues of the operator $T^{1/2}JT^{1/2}$, numbered with multiplicities taken into account in decreasing order of absolute values, then

$$W(\lambda) = \int_0^1 \left(E + \frac{2i}{\lambda - \varphi(x)}\, dF(x)J\right), \tag{2}$$

*

The class of completely continuous self-adjoint operators satisfying condition (1) was investigated by V. I. Matsaev. The proof of Theorem 1 is based on the author' s work ([6]), in which the results of V. I. Matsaev found an essential application.

where $\varphi(x)$ is a left-continuous nondecreasing scalar function, and $F(x)$ is a strictly increasing absolutely continuous operator-function with completely continuous values, normalized by the condition $F(0) = 0$. The integral products

$$\left(E + \frac{2i}{\lambda - \varphi(\xi_1)}\Delta F_1 J\right)\left(E + \frac{2i}{\lambda - \varphi(\xi_2)}\Delta F_2 J\right)\cdots$$
$$\cdots\left(E + \frac{2i}{\lambda - \varphi(\xi_n)}\Delta F_{nJ}\right) \tag{3}$$

$$(0 = x_0 < \xi_1 \leq x_1 < \xi_2 \leq x_2 < \cdots < \xi_n \leq x_n = 1, \qquad \Delta F_k = F(x_k) - F(x_{k-1}))$$

converge in norm to $W(\lambda)$ in the sense of C. O. Shatunovskii.

Theorem 2. If $W(\lambda)$ is an entire function of $1/\lambda$ possessing properties II and III, then

$$W(\lambda) = \int_0^1 \left(E + \frac{2i}{\lambda}\, dF(x)J\right), \tag{4}$$

where $F(x)$ is a strictly increasing absolutely continuous operator-function with completely continuous values, normalized by the condition $F(0) = 0$. The integral products

$$\left(E + \frac{2i}{\lambda}\Delta F_1 J\right)\left(E + \frac{2i}{\lambda}\Delta F_2 J\right)\cdots\left(E + \frac{2i}{\lambda}\Delta F_{nJ}\right) \tag{5}$$

$$(0 = x_0 \leq \xi_1 \leq x_1 \leq \xi_2 \leq x_2 \leq \cdots \leq \xi_n \leq x_n = 1, \qquad \Delta F_k = F(x_k) - F(x_{k-1})$$

converge in norm to $W(\lambda)$ as $\max(x_k - x_{k-1}) \to 0$.

2. Let us briefly dwell on the considerations which lead to the proof of Theorem 1. Generalizing the method of M. S. Livshits ([3]), we introduce the function

$$V(\lambda) = i(W(\lambda) + E)^{-1}(W(\lambda) - E)J.$$

It is easy to verify that it is holomorphic in the upper half-plane and has non-negative imaginary part

$$\frac{V(\lambda) - V^*(\lambda)}{2i},$$

equal to zero at real points lying outside some interval (a, b). Since, moreover, $\lambda V(\lambda)$ tends to a bounded operator as $\lambda \to \infty$, there exists a nondecreasing function $\sigma(x)$ with completely continuous values such that ([5])

$$V(\lambda) = \int_a^b \frac{d\sigma(t)}{t - \lambda} \qquad (\operatorname{Im} \lambda \neq 0,\ \sigma(0) = 0).$$

Modifying somewhat the proof of a known theorem of M. A. Naimark, we represent the function $\sigma(t)$ in the form $\sigma(t) = R^*\mathcal{E}(t)R$, where R is a completely continuous mapping of the space $\mathfrak{H}$ into some Hilbert space $\mathfrak{H}_0$, and $\mathcal{E}(t)$ is an orthogonal resolution of the identity in $\mathfrak{H}_0$. Consider in $\mathfrak{H}_0$ the operator

$$A_0 = \int_a^b t\, d\mathcal{E}(t) + iRJR^*. \tag{6}$$

Analysis of formula (6) shows that the spectrum of the operator A_0 is the set of singular points of the function $W(\lambda)$. Direct computation of the resolvent of the operator A_0 leads to the formula

$$W(\lambda) = E - 2iR^*(A_0 - \lambda E)^{-1}RJ. \tag{7}$$

In an arbitrary Hilbert space $\mathfrak{H}_1$ define a Hermitian operator A_1 so that its spectrum is a part of the spectrum of the operator A_0,

and introduce the operator $A = A_0 \oplus A_1$. We then extend the domain of definition of the operator R^* to the whole space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, assuming that R maps $\mathfrak{H}_W$ into $\mathfrak{H}$.

It is obvious that the operator A has a completely continuous imaginary part

$$\frac{A - A^*}{2i} = RJR^*$$

and a purely real spectrum. Since $T = R^*R$, the eigenvalues different from zero of the operator

$$\frac{A - A^*}{2i}$$

are eigenvalues of the operator $T^{1/2}JT^{1/2}$ and therefore satisfy condition (1). From the listed properties of the operator A it follows that it admits a triangular representation [6]. By a special choice of the operator A_1, this representation can be written in the form

$$A = \int_0^1 \varphi(x)\, dE(x) + 2i \int_0^1 E(x)K\, dE(x) \quad \left(K = \frac{A - A^*}{2i}\right), \tag{8}$$

where $\varphi(x)$ is a left-continuous nondecreasing function, and $E(x)$ is such an absolutely continuous orthogonal resolution of the identity that the function $R^*E(x)R$ is strictly increasing. The spectrum $\mathfrak{N}$ of the operator (8) coincides with the set of singular points of the function $W(\lambda)$ and with the closure of the set of values of the function $\varphi(x)$ $(0 < x \leqslant 1)$, and the integral sums

$$A_\Delta = \sum_{k=1}^{n} \varphi(\xi_k)\Delta E_k + 2i \sum_{r<s} \Delta E_{rK} \Delta E_s$$

$$(0 = x_0 < x_1 < \cdots < x_n = 1; \qquad x_{k-1} < \xi_k \leqslant x_k; \quad \Delta E_k = E(x_k) - E(x_{k-1}))$$

converge in norm to A in the sense of S. O. Shatunovskii.

If $\lambda \bar{\in} \mathfrak{N}$, then

$$\left(\sum_{r<s} \frac{\Delta E_{rK} \Delta E_s}{\varphi(\xi_s) - \lambda}\right)^n = 0$$

and, consequently,

$$(A_\Delta - \lambda E)^{-1} = \sum_{s=1}^{n} \frac{\Delta E_s}{\varphi(\xi_s) - \lambda} \left(E + 2i \sum_{r<s} \frac{\Delta E_{rK} \Delta E_s}{\varphi(\xi_s) - \lambda}\right)^{-1} =$$

$$= -\sum_{r=1}^{n} \frac{\Delta E_r}{\lambda - \varphi(\xi_r)} - 2i \sum_{r<s} \frac{\Delta E_r}{\lambda - \varphi(\xi_r)} RJR^* \frac{\Delta E_s}{\lambda - \varphi(\xi_s)} -$$

$$-(2i)^2 \sum_{r<s<t} \frac{\Delta E_r}{\lambda - \varphi(\xi_r)} RJR^* \frac{\Delta E_s}{\lambda - \varphi(\xi_s)} RJR^* \frac{\Delta E_t}{\lambda - \varphi(\xi_t)} - \cdots$$

$$\cdots - (2i)^{n-1} \frac{\Delta E_1}{\lambda - \varphi(\xi_1)} RJR^* \frac{\Delta E_2}{\lambda - \varphi(\xi_2)} RJR^* \cdots \frac{\Delta E_n}{\lambda - \varphi(\xi_n)}.$$

Thus,

$$E - 2iR^*(A_\Delta - \lambda E)^{-1} RJ = \prod_{k=1}^{n} \left(E + 2i \frac{R^* \Delta E_{kRJ}}{\lambda - \varphi(\xi_k)} \right). \tag{9}$$

Since

$$W(\lambda) = E - 2iR^*(A - \lambda E)^{-1} RJ,$$

then, putting $F(x) = R^* E(x) R$ and passing to the limit in equality (9), we obtain formula (2).

Theorem 2 is proved analogously. The only difference is that the operator A_0 in this case is completely continuous, since its imaginary part is completely continuous and its spectrum consists of only one point, 0. In connection

With this, condition (1) becomes superfluous, and (7) and (8) are replaced by the formula

$$A = 2i \int_0^1 E(x) K \, dE(x) \qquad \left(K = \frac{A - A^*}{2i} \right). \tag{10}$$

3. Let us also note the following relations. Under the conditions of Theorem 1 there exists the integral

$$W(x, y, \lambda) = \int_x^y \left(E + \frac{2i}{\lambda - \varphi(t)} \, dF(t) J \right) \qquad (x < y),$$

which, for arbitrary fixed x and y, satisfies conditions I, II, III, and (1). The function $W(x, y, \lambda)$ satisfies the integral equation

$$W(x, y, \lambda) = E + 2i \int_x^y \frac{W(x, t, \lambda) \, dF(t) J}{\lambda - \varphi(t)}$$

and expands into a norm-convergent series

$$W(x, y, \lambda) = E + 2i \int_x^y \frac{dF(t)J}{\lambda - \varphi(t)} + (2i)^2 \int_x^y \int_x^{x_1} \frac{dF(t)J}{\lambda - \varphi(t)} \frac{dF(x_1)J}{\lambda - \varphi(x_1)} + (2i)^3 \int_x^y \int_x^{x_2} \int_x^{x_1} \frac{dF(t)J}{\lambda - \varphi(t)} \frac{dF(x_1)J}{\lambda - \varphi(x_1)} \frac{dF(x_2)J}{\lambda - \varphi(x_2)} + \cdots.$$

Moreover,

$$W(x,y,\lambda)W(y,z,\lambda) = W(x,z,\lambda) \qquad (x<y<z),$$

$$W(x,y,\lambda)JW^*(x,y,\mu) - J = 2i(\overline{\mu}-\lambda)\int_x^y \frac{W(x,t,\lambda)\,dF(t)\,W^*(x,t,\mu)}{(\lambda-\varphi(t))(\overline{\mu}-\varphi(t))}.$$

Analogous relations hold under the conditions of Theorem 2.

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CITED LITERATURE

1. V. P. Potapov, DAN, **72**, No. 5, 849 (1950).
2. V. P. Potapov, Tr. Moscow Math. Soc., **4**, 125 (1955).
3. M. S. Livshits, Mat. Sbornik, **34** (76), 145 (1954).
4. Yu. P. Ginzburg, DAN, **117**, No. 2, 171 (1957).
5. N. I. Akhiezer, I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Moscow-Leningrad, 1950.
6. M. S. Brodskii, DAN, **133**, No. 6, 1271 (1960).
7. M. S. Brodskii, UMN, **16**, issue 1 (97) (1961).

Note: Figure translations are in progress. See original paper for figures.