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Abstract

Full Text

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GEOMETRIC DESCRIPTION OF ALL POSSIBLE REPRESENTATIONS OF A RIEMANNIAN METRIC IN LEVI-CIVITA FORM

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1°. As is known, the problem of the geodesic mapping of Riemannian spaces leads to metrics of the form (see ([1]))

$$ds^* = \Pi'_\alpha|f_\alpha - f_1|\, ds_1^2 + \Pi'_\alpha|f_\alpha - f_2|\, ds_2^2 + \ldots + \Pi'_\alpha|f_\alpha - f_p|\, ds_p^2, \qquad (1)$$

which are called **Levi-Civita metrics**. Here an important role is played by the associated metric

$$ds^2 = \Pi'_\alpha|f_\alpha - f_1|\, dy_1^2 + \Pi'_\alpha|f_\alpha - f_2|\, dy_2^2 + \ldots + \Pi'_\alpha|f_\alpha - f_p|\, dy_p^2.$$

If ds^{*2} does not have constant curvature, then the representation of ds^2 in Levi-Civita form is possible in only one way. If ds^{*2} has constant curvature K, the metric (1) is called **exceptional**; for exceptional metrics, representation in Levi-Civita form is possible, generally speaking, in an infinite set of ways. The purpose of the present article is to give an effective description of all representations of an exceptional metric in Levi-Civita form. For lack of space we shall confine ourselves to the most interesting case $K < 0$.

2°. Suppose, for definiteness, that

$$f_1(x_1), \ldots, f_r(x_r) \neq \text{const}, \qquad f_{r+1}, \ldots, f_p = \text{const} \quad (0 < r < p). \qquad (2)$$

Writing ds^2 in the form

$$ds^2 = ds_0^2 + \varphi_{r+1} ds_{r+1}^2 + \ldots + \varphi_p ds_p^2 \quad (\dim ds_0^2 = r) \qquad (3)$$

(where ds_0^2 denotes the sum of the first r terms in (1)), we see that ds_0^2 defines a space $V(K)$, and (3) is a K-decomposition of the metric ds^2.

Thus the problem posed splits into two:

I. To describe all K-decompositions in the given space $V(K)$.

II. For each of such K-decompositions (3), to indicate all ways of representing it in the form (1); in fact, what is required here is to find certain special r-orthogonal coordinate systems in the space of constant curvature ds_0^2.

3°. We shall describe certain special coordinate systems in the Lobachevskii space L_r of curvature -1.

A. Fix a certain plane L_{r_0-1}. Through an arbitrary point $M \in L_r$ draw the plane L'_{r_0} containing L_{r_0-1}. As coordinates of the point M we take: a) the coordinates of the point M_0, the projection of M onto L_{r_0-1}, and also the number $\tau = MM_0$; b) the angular coordinates of the plane L'_{r_0} itself. This is a generalized polar coordinate system with axis (more precisely, "pole") L_{r_0-1}. The metric of L_r in such a system has the form $ds_0^2 + \varphi\, d\lambda_0^2$, where ds_0^2 is the metric in L'_{r_0}.

The following items B, C, D exhaust all coordinate systems in which the metric of L_r is reduced to the form

$$dS_0^2 = ds_0^2 + \varphi\, d\lambda_0^2 + \psi\, d\mu_0^2, \tag{4}$$

where the functions φ, ψ depend only on the variables x^s.*

B. Two planes $\widetilde{L}$, $\widetilde{\widetilde{L}}$ of dimensions $r_0 + r_1 - 1$ and $r_0 + r_2 - 1$, respectively, are fixed, intersecting orthogonally in some plane L^*. Through an arbitrary point M pass the plane $L'_{r_0+r_1} \supset \widetilde{L}$ and the plane $L'_{r_0+r_2} \supset \widetilde{\widetilde{L}}$. As the coordinates of the point M one takes: a) the coordinates of the point M^*—the projection of M onto L^*—in L^*, and also the numbers $\tau_1 = M\widetilde{M}$, $\tau_2 = M\widetilde{\widetilde{M}}$; b) the angular coordinates of the plane $L'_{r_0+r_1}$ (x^{μ_0}); c) the angular coordinates of the plane $L'_{r_0+r_2}$ (x^{λ_0}). In this case $\varphi = \operatorname{sh}^2\tau_1$, $\psi = \operatorname{sh}^2\tau_2$.

C. A plane $\widetilde{L}$ of dimension r_2 is fixed. At each point $\widetilde{M} \in \widetilde{L}$ a plane L' is constructed—the complete normal to $\widetilde{L}$. In each plane L' a generalized polar coordinate system is introduced with axis L'_{r_0-1} passing through $\widetilde{M}$; all such coordinate systems (in the various L') are coordinated in a natural way. As the coordinates of a point $M \in L'$ one takes: a) the coordinates of the point $\widetilde{M}$ in $\widetilde{L}$ (x^{μ_0}); b) the polar coordinates of the point M in L'. Instead of specifying at each point $\widetilde{M}$ the axis L'_{r_0-1}, one may specify one plane $\widetilde{\widetilde{L}} = \widetilde{L} \dot{+} L'_{r_0-1}$. In this case $\varphi = \operatorname{sh}^2\tau_1$ (τ_1 is the distance to the axis), $\psi = \operatorname{ch}^2\tau_2$, where $\tau_2 = M\widetilde{M}$.

D. An orisphere O of dimension r_2 is fixed. At each point $\widetilde{M} \in O$ a plane L' is constructed—the complete normal to O. Obviously, L' contains the line l'—the axis of the orisphere at the point $\widetilde{M}$. In L' a polar coordinate system is introduced with axis $L'_{r_0-1} \supset l'$; all such systems are coordinated in a natural way. As the coordinates of a point $M \in L'$ one takes: a) the coordinates of the point $\widetilde{M}$ in O (x^{μ_0}); b) the polar coordinates of the point M in L'. In this case

$\varphi = \operatorname{sh}^2 \tau_1$, $\psi = e^{2\tau_2}$, where τ_2 is the distance to the hyperorisphere containing O. In general the coordinate system is determined by a pair of planes $\widetilde{L}$, $\widetilde{\widetilde{L}}$ ($\widetilde{L} = L_{r_2+1} \supset O$, $\widetilde{\widetilde{L}} = O \dot{+} L'_{r_0-1}$) and a line l belonging to both planes.

4°. **Solution of Problem I.** We describe here all K-decompositions of the form

$$ds^2 = ds_0^2 + \varphi\, dS_1^2 + \psi\, dS_2^2 \tag{5}$$

(the number of forms dS_α^2 is equal to two). Let

$$ds^2 = dS_0^2 + F_1 ds_1^2 + \dots + F_{qds} q^2 \tag{6}$$

be a maximal K-decomposition. The problem reduces to finding, in the space dS_0^2 of constant curvature K, all coordinate systems of the form (4) which in a certain way "split" the functions $F_1, \dots, F_q$; namely, some of them lead to the form $\Phi_\alpha = \varphi\rho_\alpha$, and the other part to the form $\Psi_\beta = \psi\sigma_\beta$, where the functions ρ_α depend only on the coordinates x^{λ_0}, and the functions σ_β on x^{μ_0}. Indeed, then we shall have (5), where

$$dS_1^2 = d\lambda_0^2 + \sum_\alpha \rho_\alpha\, d\lambda_\alpha^2, \qquad dS_2^2 = d\mu_0^2 + \sum_\beta \sigma_\beta\, d\mu_\beta^2$$

(the forms $d\lambda_\alpha^2, d\mu_\beta^2$ coincide, in some order, with $ds_1^2, \dots, ds_q^2$).

We regard (6) as given. For $K = -1$ this decomposition has one of the forms:

$$ds^2 = dS_0^2 + y_{r+1}^2 ds_1^2 + y_1^2 ds_2^2 + \dots + y_{q-1}^2 ds_q^2 \quad (q \le r+1); \tag{7}$$

$$ds^2 = dS_0^2 + (y_{r+1} - y_r)^2 ds_1^2 + y_1^2 ds_2^2 + \dots + y_{q-1}^2 ds_q^2 \quad (q \le r); \tag{8}$$

$$ds^2 = dS_0^2 + y_1^2 ds_1^2 + y_2^2 ds_2^2 + \dots + y_q^2 ds_q^2 \quad (q \le r), \tag{9}$$

where dS_0^2 is the metric of the imaginary unit sphere $y_1^2 + \dots + y_r^2 - y_{r+1}^2 = -1$ in the pseudo-Euclidean space $E_{r+1}^{(1)}$: $dl^2 = dy_1^2 + \dots + dy_r^2 - dy_{r+1}^2$.

$^* x^{s_0}$ are coordinates in ds_0^2, $\quad x^{\lambda_0}$ in $d\lambda_0^2$, $\quad x^{\mu_0}$ in $d\mu_0^2$.

We also put $r_0 = \dim ds_0^2$, $r_1 = \dim d\lambda_0^2$, $r_2 = \dim d\mu_0^2$.

We identify the sphere indicated above with Lobachevsky space L_r. All possible planes in L_r are sections of the sphere by subspaces of the space $E_{r+1}^{(1)\,*}$.

In case (7), all coordinate systems (4) of interest to us are obtained as follows: we divide the coordinates $y_{r+1}, y_1, \dots, y_{q-1}$ into two groups, for example, $y_{r+1}, y_1, \dots, y_k$ and $y_{k+1}, \dots, y_{q-1}$; the coordinate system (4) is determined by scheme C, where $\tilde{\tilde{L}}$ is the section of L_r by the subspace contained in $y_{k+1} = \dots = y_{q-1} = 0$, and $\tilde{L}$ is the section of L_r by the subspace containing the axes $Oy_{r+1}, Oy_1, \dots, Oy_k$; moreover, it is required that $\tilde{L} \subset \tilde{\tilde{L}}$.

In case (8), all coordinate systems (4) of interest to us are obtained as follows: we divide the coordinates $y_1, \dots, y_{q-1}$ into two groups, for example $y_1, \dots, y_k$ and $y_{k+1}, \dots, y_{q-1}$; the coordinate system (4) is determined by scheme C, where $\tilde{\tilde{L}}$ is determined as above, and $\tilde{L}$ is the section of L_r by the subspace containing the axes $Oy_1, \dots, Oy_k$ and the line $y_1 = \dots = y_{r-1} = 0,\ y_r = y_{r+1}$; moreover, it is required that $\tilde{L} \subset \tilde{\tilde{L}}$.

In case (9), coordinate systems of all types B, C, D are possible. As above, we divide the coordinates $y_1, \dots, y_q$ into two groups, for example $y_1, \dots, y_k$ and $y_{k+1}, \dots, y_q$. For scheme B one takes the planes $\tilde{L}, \tilde{\tilde{L}}$ contained in the subspaces $y_1 = \dots = y_k = 0$ and $y_{k+1} = \dots = y_q = 0$, respectively; moreover, it is required that $\tilde{L}$ and $\tilde{\tilde{L}}$ intersect (generally speaking, in some plane L^s) orthogonally. For scheme C one takes the plane $\tilde{\tilde{L}}$ lying in the subspace $y_{k+1} = \dots = y_q = 0$, and $\tilde{L}$ in the subspace containing the axes $Oy_1, \dots, Oy_k$; moreover, $\tilde{L} \subset \tilde{\tilde{L}}$. For scheme D one takes a line l lying in the subspace $y_1 = \dots = y_q = 0$, the plane $\tilde{\tilde{L}}$ in the subspace $y_{k+1} = \dots = y_q = 0$, and the plane $\tilde{L}$ in the subspace containing the axes $Oy_1, \dots, Oy_k$.

5°. Solution of Problem II. Consider the exceptional metric (1). Without loss of generality, to conditions (2) one may add $f_1 < f_2 < \dots < f_r$. Passing to new coordinates $\rho_1 = f_1, \dots, \rho_r = f_r$ brings ds_0^2 to the form

$$ds_0^2 = \sum_{i=1}^{r} \frac{1}{P(\rho_i)} \Pi_i'(\rho_1 - \rho_i)\, d\rho_i^2, \tag{10}$$

where $P(\rho)$ is a polynomial among whose roots all the numbers $f_{r+1}, \dots, f_p$ are contained. For $K = -1$ the polynomial $P(\rho)$ has the form $(-1)^r(\rho - \alpha_1) \cdots (\rho - \alpha_{r-1})(\rho - \beta)(\rho - \gamma)$, $\alpha_1 < \alpha_2 < \dots < \alpha_{r-1}$; for β and γ there are the various possibilities listed in items 1)–6). Thus, starting from a given K-decomposition (3), we must solve two problems:

1. Find all coordinate systems in ds_0^2 in which (10) occurs. For this purpose we shall use [2], where the required coordinate systems were found for $r = 3$; for $r > 3$ everything is analogous.
2. Among these coordinate systems select those for which

$$\varphi_{r+1} = A_{r+1}\prod_{i=1}^{r}(\rho_i - f_{r+1}), \dots, \varphi_p = A_p\prod_{i=1}^{r}(\rho_i - f_p),$$

where $A_{r+1}, \dots, A_p$ are constants, and $f_{r+1}, \dots, f_p$ are part of the roots of $P(\rho)$.

* Containing at least one vector with negative scalar square.

Let us list all the solutions of problem 1. Recall that ds_0^2 is the metric on the imaginary unit sphere in $E_{r+1}^{(1)}$.

1) $\alpha_{r-1} < \beta < \gamma$. Let $z_1, \dots, z_{r+1}$ be any Cartesian coordinates* in $E_{r+1}^{(1)}$. Consider the equation with respect to ρ:

$$\frac{z_1^2}{\rho - \alpha_1} + \dots + \frac{z_{r-1}^2}{\rho - \alpha_{r-1}} + \frac{z_r^2}{\rho - \beta} - \frac{z_{r+1}^2}{\rho - \gamma} = 0. \tag{11}$$

Its roots satisfy the condition $\rho_1 < \alpha_1 < \rho_2 < \dots < \alpha_{r-1} < \rho_r < \beta < \gamma$. Taking $\rho_1, \dots, \rho_r$ as new coordinates in ds_0^2, we obtain (10).

2) $\alpha_{k-1} < \beta < \gamma < \alpha_k$ $(1 < k \leq r-1)$. Equation (11).
3) $\beta = a + bi, \quad \gamma = a - bi$. Equation (11) with z_r replaced by $\tilde{z}_r = \frac{1}{\sqrt{2}}(z_r - iz_{r+1})$ and z_{r+1} by $\tilde{z}_{r+1} = \frac{i}{\sqrt{2}}(z_r + iz_{r+1})$.
4) $\alpha_{r-1} < \beta = \gamma$. The equation

$$\frac{z_1^2}{\rho - \alpha_1} + \dots + \frac{z_{r-1}^2}{\rho - \alpha_{r-1}} + \frac{z_r^2}{\rho - \beta} - \frac{z_{r+1}^2}{\rho - \beta} - \frac{(z_r - z_{r+1})^2}{(\rho - \beta)^2} = 0. \tag{12}$$

5) $\alpha_{k-1} < \beta = \gamma < \alpha_k$. Equation (12).
6) $\beta = \gamma = \alpha_k$. The equation

$$\frac{z_1^2}{\rho - \alpha_1} + \dots + \frac{z_{r-1}^2}{\rho - \alpha_{r-1}} + \frac{z_r^2}{\rho - \alpha_k} - \frac{z_{r+1}^2}{\rho - \alpha_k} - \frac{2z_k(z_r - z_{r+1})}{(\rho - \alpha_k)^2} + \frac{(z_r - z_{r+1})^2}{(\rho - \alpha_k)^3} = 0.$$

Let us list all the solutions of problem 2. First of all, in $E_{r+1}^{(1)}$ we fix a Cartesian coordinate system that brings the given K-decomposition (3) to one of the forms (7), (8), (9). All the desired coordinate systems $\rho_1, \dots, \rho_r$ in ds_0^2 are obtained from the admissible Cartesian coordinate systems in $E_{r+1}^{(1)}$ listed below according to the rules indicated in 1)–6).

Case (9). 1)–2)** We fix an (ordered) group of $q = r - p$ roots of the polynomial $P(\rho)$, excluding γ: $f_{r+1} = \alpha_{i_1}, \dots, f_p = \alpha_{i_q}$. We perform the corresponding renumbering of the coordinates y: $y_1 \to y_{i_1}, \dots, \dots, y_q \to y_{i_q}$. Those coordinate systems $z_1, \dots, z_{r+1}$ are regarded as admissible in which the axes $Oz_{i_1}, \dots, Oz_{i_q}$

coincide with the axes $Oy_{i_1}, \dots, Oy_{i_q}$ (respectively). If one of the selected roots, for example f_{r+1}, is β, then we set $i_1 = r$. 3)–6) We fix a group of q roots, excluding β, γ. Thereafter everything is as in 1)–2).

Case (7). 1)–2) We fix a group of q roots, with the first being γ: $f_{r+1} = \gamma$, $f_{r+2} = \alpha_{i_1}, \dots, f_p = \alpha_{i_{q-1}}$. We perform the renumbering: $y_1 \to y_{i_1}, \dots, y_{q-1} \to y_{i_{q-1}}$. Those systems are regarded as admissible in which the axes $Oz_{r+1}, Oz_{i_1}, \dots, Oz_{i_{q-1}}$ coincide with the axes $Oy_{r+1}, Oy_{i_1}, \dots, Oy_{i_{q-1}}$ (respectively).

Case (8). 4)–5) Everything is as in the preceding case, but instead of coincidence of the axes Oz_{r+1}, Oy_{r+1} one requires coincidence of the planes $Oz_{rz_{r+1}}$ and $Oy_{ry_{r+1}}$. 6) We fix the group $f_{r+1} = \alpha_k$, $f_{r+2} = \alpha_{i_1}, \dots, f_p = \alpha_{i_{q-1}}$. Thereafter everything is as in the preceding case.

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* We use only such Cartesian coordinate systems in $E^{(1)}_{r+1}$ in which the metric is $dz_1^2 + \dots + dz_r^2 - dz_{r+1}^2$ (the minus sign stands before the last square).

** Below follows a listing of admissible coordinate systems for items 1)–2). The subsequent numbering has an analogous meaning.

Note: Figure translations are in progress. See original paper for figures.