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Abstract

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THE SCATTERING PROBLEM FOR A DIFFERENCE EQUATION

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In the present note the direct and inverse scattering problems are set forth for a difference equation with operator coefficients. The method for solving the problem is approximately the same as that used for the Schrödinger equation ([2,4,7]). For clarity we shall first present the scattering problem for the classical difference equation, and then formulate analogous results for a difference equation with operator coefficients.

1°. Consider the difference equation

$$\frac{1}{2}\left(u_{j+1}+u_{j-1}\right)+c_j u_j=\lambda u_j; \tag{1}$$

where u_j $(-\infty < j < +\infty)$ is the unknown sequence of complex numbers, $|\lambda|<1$, $\operatorname{Im}\lambda=0$. Suppose that the sequence c_j is real and that $c_j=O(1/|j|^{1+\varepsilon})$ $(\varepsilon>0)$ as $|j|\to\infty$. Under these conditions the difference expression standing on the left-hand side of equation (1) generates a bounded self-adjoint operator in l_2 ([1]). From the condition that c_j decreases at infinity it follows that all solutions of equation (1) are bounded.

The scattering problem for equation (1) is formulated as follows:

Find a solution of equation (1) having the form

$$u_j=e^{\nu ij\theta}+v_j^{(\nu)}(\lambda) \tag{2}$$

$(\theta=\arccos\lambda,\ \nu=\pm1)$, *where* $v_j^{(\nu)}(\lambda)$ *satisfies the conditions*

$$v_{j+1}^{(\nu)}(\lambda)-e^{-i\theta}v_j^{(\nu)}(\lambda)=O(1)\quad\text{as } j\to+\infty,$$

$$v_{j+1}^{(\nu)}(\lambda)-e^{i\theta}v_j^{(\nu)}(\lambda)=O(1)\quad\text{as } j\to-\infty. \tag{3}$$

We shall call conditions (3) the **radiation conditions**.*

Let us clarify the physical meaning of the problem posed. Consider an unbounded one-dimensional lattice consisting of identical particles of mass m, arranged along a straight line and, in the equilibrium position, located at equal distances d from one another. We shall assume that only neighboring particles interact with one another and that the force of this interaction depends only on their mutual distance. Suppose, moreover, that external elastic forces, independent of time, act on the particles. The equation of longitudinal oscillations of such a lattice is written as follows (^{6}):

$$m\frac{d^2x_j}{dt^2} = (x_{j+1} + x_{j-1} - 2x_j)U''(d) + k_j x_j; \tag{4}$$

where x_j is the displacement of the j-th point from its equilibrium position; t is time; $U(x)$ is the potential energy of interaction of two particles located at a distance x. By separation of variables, equation (4) is reduced to a stationary equation of the form (1). It is easy to verify that $u_j = e^{\nu ij\theta}$ ($\theta = \arg\cos\lambda,\ \nu = \pm 1$) satisfies the equation obtained from (1) for $c_j = 0$ for all j. Thus, the first term in (2) characterizes a wave propagating in a free lattice, while the second term is the scattering effect caused by the presence of additional external forces.

Let us proceed to the solution of problem (1)–(3). We note that the spectrum of the bounded self-adjoint operator generated in l_2 by the difference expression

$$\frac{1}{2}(u_{j+1} + u_{j-1}) \quad (-\infty < j < +\infty)$$

is the segment $[-1, 1]$. This operator has no eigenvalues. The limit, as $z \to \lambda$ ($\operatorname{Im} z > 0,\ \operatorname{Im}\lambda = 0,\ |\lambda| < 1$), of the resolvent matrix of this operator is equal to

$$\frac{i}{\sin\theta}\, e^{-i|j-k|\theta}, \quad \text{where } \theta = \arccos\lambda.$$

Denote by Φ the Banach space whose elements are sequences of complex numbers, with norm

$$\|f\| = \sup_j \rho_j |f_j|, \tag{5}$$

where the "weight" sequence ρ_j consists of positive numbers and satisfies the condition $\rho_j = O(1/|j|^\alpha)$ ($0 < \alpha < \varepsilon$) as $|j| \to \infty$. We shall further denote by K the operator in Φ acting according to the formula

$$(Kf)_j = -\frac{i}{\sin\theta} \sum_{k=-\infty}^{\infty} e^{-i|j-k|\theta} c_k f_k. \tag{6}$$

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* In the case of the Schrödinger equation, in addition to conditions analogous to (3), one more condition appears; for the one-dimensional case the latter becomes a boundedness condition, which, by virtue of the remark made above, is satisfied automatically.

Lemma 1. The scattering problem for equation (1) is equivalent to the problem of solving the equation

$$v = Kv + K\left(e^{\nu ij\theta}\right) \tag{7}$$

in the space Φ.

Lemma 2. The operator K in the space Φ is a completely continuous operator.

Lemma 3. Solutions of the equation $f = Kf$ from the space Φ belong to l_2.

If we take into account that the operator generated in l_2 by the difference expression standing on the left-hand side of (2) has no discrete spectrum in the interval $(-1, 1)$, then from Lemmas 1-3 it follows:

Theorem 1. The scattering problem for equation (1) has one and only one solution.

If by $R^{+}_{\lambda;j,k}$ we denote the limit, as $z \to \lambda$ $(\operatorname{Im} z > 0,\ \operatorname{Im}\lambda = 0,\ |\lambda| < 1)$, of the resolvent matrix of the difference operator generated in l_2 by the difference expression standing on the left-hand side of (1), then the solution of the scattering problem can be written in the form

$$v_j^{(\nu)}(\lambda) = -\sum_{m=-\infty}^{\infty} R^{+}_{\lambda;j,m} c_m e^{\nu i m\theta}. \tag{8}$$

We also record a formula that will be needed in what follows: for $|\lambda| < 1$,

$$\frac{d}{d\lambda} E_{\lambda;j,k} = \frac{1}{\pi} \operatorname{Im} R^{+}_{\lambda;j,k}, \tag{9}$$

where $E_{\lambda;j,k}$ is the matrix of the resolution of the identity of the difference operator under consideration.

2°. In this subsection, in order to simplify the exposition, we impose on c_j a stronger restriction: let $c_j = 0$ for $|j| \geq n_0$. Then from (8) one can obtain that, for $|k| > n_0$, the formula

$$R^{+}_{\lambda;j,k} = \frac{i}{\sin\theta}\left[e^{-i|j-k|\theta} + e^{-i|k|\theta} v_j^{(\nu)}(\lambda)\right], \tag{10}$$

holds, where $\nu = 1$ for $k > n_0$ and $\nu = -1$ for $k < -n_0$.

From (10) and (9) it follows that, knowing $v_n^{(1)}(\lambda)$ and $v_{n+1}^{(1)}(\lambda)$ $(n > n_0)$, one can reconstruct the spectral matrix $(^3)$ of problem (1) for $|\lambda| < 1$. If, in addition, the jumps of the spectral matrix for $|\lambda| \geq 1$ are known, then the spectral matrix is known for all λ, and this, as is known, uniquely determines the coefficients of equation (1).

Thus we obtain:

Theorem 2. *If the scattered wave $v_j^{(1)}(\lambda)$ is known at two points $j = n,\ n+1$ $(n > n_0)$ for all $\lambda,\ |\lambda| < 1$, and the jumps of the spectral matrix for $|\lambda| \geq 1$ are known, then the problem of reconstructing equation (1) is uniquely solvable.*

3°. We now pass to the study of the scattering problem for a difference equation with operator coefficients. Let H be a Hilbert space in which some involution is given. Consider the difference equation

$$\frac{1}{2}(u_{j+1} + u_{j-1}) + C_j u_j = \lambda u_j^* \tag{11}$$

$(-\infty < j < +\infty)$, where $u_j \in H,\ |\lambda| < 1$, and C_j are bounded self-adjoint and real (i.e., preserving the involution) operators in H. In addition, let $C_j = 0$ for $|j| \geq n_0$.* Under these conditions the difference expression standing on the left-hand side of (11) generates in $l_2(H)$ a bounded self-adjoint operator (^{5}).

We formulate the scattering problem for equation (11):

For a given $x \in H,\ \|x\| = 1$, find a solution of equation (11) having the form

$$u_j = x e^{\nu i j \theta} + v_j^{(\nu)}(\lambda; x) \tag{12}$$

$(\theta = \arg\cos\lambda;\ \nu = \pm 1)$, *where the sequence of elements $v_j^{(\nu)}(\lambda; x) \in H$ satisfies the radiation conditions:*

$$\begin{aligned} \left\| v_{j+1}^{(\nu)}(\lambda; x) - e^{-i\theta} v_j^{(\nu)}(\lambda; x) \right\| &= o(1) \quad \text{as } j \to +\infty, \\ \left\| v_{j+1}^{(\nu)}(\lambda; x) - e^{i\theta} v_j^{(\nu)}(\lambda; x) \right\| &= o(1) \quad \text{as } j \to -\infty. \end{aligned} \tag{13}$$

Denote by Ψ the Banach space whose elements are sequences of elements from H with norm

$$\|f\|_1 = \sup_j \|A_j f_j\|, \tag{14}$$

where the "weight" sequence of bounded operators A_j in H has the following properties: the operator A_j, for each j, is a completely continuous operator for which zero is not an eigenvalue, and $\|A_j\| = O(1/|j|)$ as $|j| \to \infty$. Finally, denote by M the operator in Ψ acting according to the formula

$$(Mf)_j = -\frac{i}{\sin\theta} \sum_{k=-\infty}^{\infty} e^{-i|j-k|\theta} C_k f_k. \tag{15}$$

* In some cases it is also possible to consider an infinite sequence of operators with rapidly decreasing norms, for example, if there exists such a completely

continuous operator A that all the operators $\|C_{jA}^{-1}\|$ are bounded and $\|C_{jA}^{-1}\| = O(1/|j|^{1+\varepsilon})$.

Lemma 4. The scattering problem for equation (11) is equivalent to the problem of solving the equation

$$v = Mv + M(xe^{\nu i\theta}) \tag{16}$$

in the space Ψ.

Lemma 5. The operator M in the space Ψ is a completely continuous operator.

Lemma 6. The solutions of the equation $f = Mf$ belonging to the space Ψ belong to $l_2(H)$.

Theorem 3. The scattering problem for equation (11) has one and only one solution.

The solution of the scattering problem (11)–(13) can be written in the form

$$v_j^{(\nu)}(\lambda; x) = -\sum_{m=-\infty}^{\infty} R^{+}_{\lambda;j,m} C_m x e^{\nu i m\theta}, \tag{17}$$

where $R^{+}_{\lambda;j,m}$ is the limit, as $z \to \lambda$ ($\operatorname{Im} z > 0$, $\operatorname{Im}\lambda = 0$, $|\lambda| < 1$), of the resolvent matrix of the difference operator generated in $l_2(H)$ by the difference expression standing on the left-hand side of (11).

4°. Let us note that in the case under consideration there is a formula analogous to (9):

$$\frac{d}{d\lambda}(E_{\lambda;j,k}x, x) = \frac{1}{\pi}\operatorname{Im}(R^{+}_{\lambda;j,k}x, x), \tag{18}$$

where $E_{\lambda;j,k}$ is the matrix of the resolution of the identity of the difference operator under consideration, $x \in H$, and also a formula analogous to (10):

$$R^{+}_{\lambda;j,k}x = \frac{i}{\sin\theta}\left[xe^{-i|j-k|\theta} + e^{-i|k|\theta} v_j^{(\nu)}(\lambda; x)\right], \tag{19}$$

where $x \in H,\ \|x\| = 1,\ |k| > n_0;\ \nu = 1$ for $k > n_0$ and $\nu = -1$ for $k < -n_0$.

From formulas (18) and (19), analogously to the preceding, it follows that

Theorem 4. If at two points $j = n, n+1$ ($n > n_0$) the scattered wave $v_j^{(1)}(\lambda, x)$ is known for all $\lambda,\ |\lambda| < 1$, and all $x \in H,\ \|x\| = 1$, and the jumps of the spectral matrix are known for $|\lambda| \geq 1$, then the problem of reconstructing equation (11) is uniquely solvable.

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Note: Figure translations are in progress. See original paper for figures.

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