ON PARTIALLY HYPOELLIPTIC EQUATIONS AND POLYNOMIALS

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Abstract

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ON PARTIALLY HYPOELLIPTIC EQUATIONS AND POLYNOMIALS

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This note contains results developing or generalizing certain theorems from ([1–3]).

We shall consider the differential operator

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right), \tag{1}$$

where

$$P(s) = P(s_1, \dots, s_n) \tag{2}$$

is a polynomial with constant complex coefficients. In what follows the following notation is used: C_n is the complex n-dimensional space,
$R_n^k = \{s \in C_n : \operatorname{Im} s_j = 0 \text{ for } j \neq k\}$;
$N(P) = \{s \in C_n : P(s) = 0\}$, $N^k(P) = N(P) \cap R_n^k$;
$s = \sigma + i\tau$; $|\sigma|^2 = \sum_{1 \le k \le n} |\sigma_k|^2$, and so on; a, b are certain positive constants. Generalized functions are linear continuous functionals on the space K of finite infinitely differentiable functions.

The polynomial (2) is called $\binom{k}{p}$-hypoelliptic (with exponent $\gamma > 0$) if on the manifold $N^k(P)$, from $|\sigma_p| \to \infty$ it follows that $|\tau| = |\tau_k| \to \infty$ (respectively, $|\tau| = |\tau_k| \ge a|\sigma_p|^\gamma - b$).

Using the theorem of Seidenberg-Tarski ([4,6]), one can show that every $\binom{k}{p}$-hypoelliptic polynomial is $\binom{k}{p}$-hypoelliptic with some positive exponent γ. One can also formulate a necessary and sufficient condition for the $\binom{k}{p}$-hypoellipticity of a polynomial in terms of its behavior for real values of the arguments, analogous to Hörmander' s condition ([1,2]) and to V. P. Palamodov' s formula for computing the genus of a polynomial ([5]). Namely, the following lemma holds:

Lemma. In order that the polynomial (2) be $\binom{k}{p}$-hypoelliptic with exponent $0 < \gamma \le 1$, it is necessary and sufficient that, for all sufficiently large in modulus σ_p, the inequality

$$\frac{|\partial P(\sigma)/\partial\sigma_k|}{|P(\sigma)|} \le \frac{a}{1+|\sigma_p|^\gamma}.$$

Theorem 1. Let the polynomial (2) be $\binom{k_1}{p_1}, \dots, \binom{k_\nu}{p_\nu}$-hypoelliptic with exponent $0 < \gamma \le 1$, where $1 = p_1 = k_1 \le \dots \le k_\nu \le n$. Then the equation

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right) E(x) = \delta(x)$$

has a solution $E(x)$ admitting the representation

$$E(x) = Q\left(\frac{1}{i}\frac{\partial}{\partial x}\right) F(x),$$

where $F(x)$ is a continuous function; moreover, in every compact set T in which $x_{k_j} \ne 0$, the function $F(x)$ has continuous derivatives of arbitrary order with respect to x_{p_j}, and

$$\max_{x\in T}\left|\frac{\partial^m F(x)}{\partial x_{p_j}^m}\right| \le ab^m \Gamma\left(\frac{m}{\gamma}\right), \qquad 1 \le j \le v. \tag{3}$$

An analogous estimate also holds for mixed derivatives, if some of the numbers k_j coincide.

Theorem 1 is proved by means of a special choice of Hörmander' s ladder (2). If, in the conditions of Theorem 1, $k_j = j,\ p_j = 1,\ 1 \le j \le v$, then by a suitable choice of the ladder one can ensure that the Fourier transform, in the sense of generalized functions, of the function $F(x)$, taken only with respect to the variables $x_{v+1}, \dots, x_n$, will be a functional of the type of an almost everywhere continuous function, for which estimates of the type (3) are preserved.

We shall call the differential operator (1) hypoelliptic with respect to the variables $x' = (x_1, \dots, x_v)$ if there exists a number $q \ge 0$ such that every solution of the equation

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right) u(x) = 0, \tag{4}$$

having continuous derivatives up to order q, is infinitely differentiable with respect to the distinguished variables.

The theorem below, Theorem 2, is a consequence of Theorem 1, the Banach closed graph theorem, and the Seidenberg–Tarski theorem.

Theorem 2. In order that the operator (1) be hypoelliptic with respect to the variables $x' = (x_1, \dots, x_v)$, it is necessary and sufficient that the polynomial (2) be $\binom{k}{p}$-hypoelliptic for all $1 \le k \le n,\ 1 \le p \le v$.

In the latter case, every sufficiently smooth solution of equation (4) locally belongs to the Gevrey class $G_{1/\gamma}$ with respect to the variables x', where $\gamma > 0$, and an arbitrary (generalized) solution locally admits the representation

$$u(x) = \sum_{m=1}^{M} P_m\left(\frac{1}{i}\frac{\partial}{\partial x''}\right) u_m(x),$$

where $u_m(x)$ are continuous functions of the Gevrey class $G_{1/\gamma}$ with respect to the variables x', and $P_m(s'')$ are polynomials depending only on the variables $s'' = (s_{v+1}, \dots, s_n)$.

We shall formulate one purely algebraic theorem which is also a consequence of Theorem 1 (an elementary proof is not known to us).

Theorem 3. Let the polynomial (2) be $\binom{1}{p}, \dots, \binom{v}{p}$-hypoelliptic with exponent $0 < \gamma \le 1$. Then the zeros of this polynomial of the form

$$s = (s_1, \dots, s_v, \sigma_{v+1}, \dots, \sigma_n)$$

satisfy the estimate

$$|\tau| = \left(\sum_{k=1}^{v} \tau_k^2\right)^{1/2} \ge a|\sigma_p|^\gamma - b.$$

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Note: Figure translations are in progress. See original paper for figures.