MATHEMATICAL PHYSICS

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Abstract

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ON A BOUNDARY-VALUE PROBLEM FOR THE LAPLACE EQUATION

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A method is proposed for determining the stationary temperature field of an unbounded hollow cylinder, the inner surface of which is covered with thermal insulation, while the outer surface over half its length has temperature

$$T = f(z),$$

whereas the other half of the outer lateral surface of the cylinder radiates heat into the surrounding medium according to Newton' s law.

The solution of the stated problem reduces to determining the temperature distribution function $T(r, z)$, satisfying the Laplace equation in a cylindrical coordinate system

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

and the boundary conditions on the lateral surfaces of the cylinder

$$\frac{\partial T}{\partial r} = 0 \quad \text{for } r = r_1,\ -\infty < z < +\infty; \tag{2}$$

$$\frac{\partial T}{\partial r} + hT = 0 \quad \text{for } r = r_2,\ 0 < z < +\infty; \tag{3}$$

$$T = f(z) \quad \text{for } r = r_2,\ -\infty < z < 0. \tag{4}$$

With respect to the function $f(z)$, assume that it can be represented on the interval $(-\infty, 0)$ by a Fourier integral, i.e.,

$$f(z) = \int_{-\infty}^{0} A(\beta) \cos \beta z \, d\beta; \tag{5}$$

$$A(\beta) = \frac{2}{\pi} \int_{-\infty}^{0} f(v) \cos \beta v \, dv. \tag{6}$$

We first find a solution of equation (1) satisfying conditions (2), (3) and the condition

$$T = A \cos \beta z = \frac{A}{2} \left(e^{i\beta z} + e^{-i\beta z} \right) \quad \text{for } r = r_2, \ -\infty < z < 0 \tag{7}$$

(A and β are real parameters).

To find this solution, consider the integral

$$T_1(\rho; \lambda) = \frac{1}{r_2} \int_C B(u) \, \frac{H_0^{(1)}(\rho u) J_1(nu) - J_0(\rho u) H_1^{(1)}(nu)}{J_1(nu)} \, e^{\lambda u} \, du, \tag{8}$$

where u is a complex parameter; C is the imaginary axis with two points $\pm i\beta r_2$ bypassed clockwise; $z = \lambda r_2$; $r = \rho r_2$; $r_1 = n r_2$.

It is not difficult to verify that the function (8), for arbitrary $B(u)$, satisfies equation (1) and condition (2), if the integral (8) together with its derivatives with respect to ρ and λ up to the second order inclusive converges absolutely and uniformly in the domain $n < \rho < 1, \ |\lambda| < \infty$.

It remains to satisfy conditions (3) and (7), which, on the basis of (8), will respectively take the form

$$\int_C k(u) e^{\lambda u} \, du = 0 \qquad \text{for } \rho = 1, \ \lambda > 0; \tag{9}$$

$$\int_C \psi(u) e^{\lambda u} \, du = \frac{A}{2} \left(e^{i\beta r_2 \lambda} + e^{-i\beta r_2 \lambda} \right) \qquad \text{for } \rho = 1, \ \lambda < 0, \tag{10}$$

where

$$k(u) = B(u) \frac{\varphi_1(u)}{r_2 J_2(nu)}, \qquad \psi(u) = \frac{\varphi_2(u)}{\varphi_1(u)} \, k(u),$$

$$\varphi_1(u) = \left[J_1(u) H_1^{(1)}(nu) - H_1^{(1)}(u) J_1(nu) \right] u +$$

$$+ h r_2 \left[H_0^{(1)}(u) J_1(nu) - J_0(u) H_1^{(1)}(nu) \right],$$

$$\varphi_2(u) = H_0^{(1)}(u) J_1(nu) - J_0(u) H_1^{(1)}(nu).$$

The boundary condition (9) will be fulfilled if $k(u)$ is regular in the domain $\mathrm{Re}(u) \leq 0$, except for the points $\pm i\beta r_2$, and satisfies in this domain the requirements of Jordan' s lemma.

The boundary condition (10) will likewise be fulfilled if $\psi(u)$ is regular in the domain $\mathrm{Re}(u) \geq 0,\ u \neq \pm i\beta r_2$, satisfies in this domain the requirements of the same lemma, and

$$\mathrm{res}\left[\psi(u)e^{\lambda u}\right]_{u=i\beta r_2} + \mathrm{res}\left[\psi(u)e^{\lambda u}\right]_{u=-i\beta r_2} = -\frac{A}{4\pi i}\left(e^{i\beta r_2\lambda} + e^{-i\beta r_2\lambda}\right).$$

Therefore

$$\mathrm{res}\left[k(u)e^{\lambda u}\right]_{u=i\beta r_2} + \mathrm{res}\left[k(u)e^{\lambda u}\right]_{u=-i\beta r_2} =$$

$$= -\frac{A\varphi_1(i\beta r_2)\left(e^{i\beta r_2\lambda} + e^{-i\beta r_2\lambda}\right)}{4\pi i\,\varphi_2(i\beta r_2)},$$

since $\varphi_1(-i\beta r_2) = -\varphi_1(i\beta r_2)$ and $\varphi_2(-i\beta r_2) = -\varphi_2(i\beta r_2)$.

To construct $k(u)$, following ([1],[2]), consider the infinite product

$$\Pi(u) = \prod_{k=1}^{\infty} \frac{1 - u/a_k}{1 - u/b_k},$$

where a_k are the positive roots of the equation

$$\varphi_1(u) = 0, \tag{11}$$

and b_k are the positive roots of the equation

$$\varphi_2(u) = 0, \tag{12}$$

where

$$\Pi(u) \sim \sqrt{\frac{-u}{n + hr_2}}$$

for sufficiently large $|u|$ in the domain $0 < \delta \leq \arg u \leq 2\pi - \delta$.

Now the function $k(u)$ can be written as follows:

$$k(u) = -\frac{A\Pi(u)\varphi_1(i\beta r_2)}{4\pi i\,\varphi_2(i\beta r_2)}\left[\frac{1}{(u - i\beta r_2)\Pi(i\beta r_2)} + \frac{1}{(u + i\beta r_2)\Pi(-i\beta r_2)}\right].$$

Taking into account that

$$\Pi(u)\Pi(-u) = \frac{\varphi_1(u)}{\varphi_2(u)}\frac{1}{n+hr_2},$$

we represent $k(u)$ in the following form:

$$k(u) = -\frac{A(n+hr_2)}{4\pi i}\Pi(u)\left[\frac{\Pi(-i\beta r_2)}{u-i\beta r_2} + \frac{\Pi(i\beta r_2)}{u+i\beta r_2}\right].$$

It is easy to establish that $k(u)$ satisfies the requirements listed above, and the function

$$T_1(\rho,\lambda) = \int_C \frac{\left[H_0^{(1)}(\rho u)J_1(nu) - J_0(\rho u)H_1^{(1)}(nu)\right]k(u)}{\varphi_1(u)}e^{\lambda u}\,du = A\varphi(\rho,\lambda,\beta), \tag{13}$$

where

$$\varphi(\rho,\lambda,\beta) = -\frac{1}{2\pi}\int_{-\infty}^{0}\left[\frac{\varphi_2(i\rho z)}{\varphi_2(iz)}\operatorname{Im}\left\{\frac{1}{\Pi(-iz)}\left[\frac{\Pi(-i\beta r_2)}{z-\beta r_2} + \frac{\Pi(i\beta r_2)}{u+\beta r_2}\right]e^{i\lambda z}\right\}\right.$$
$$\left. - \frac{\varphi_2(i\rho\beta r_2)}{\varphi_2(i\beta r_2)}\left(\frac{1}{z-\beta r_2} + \frac{1}{z+\beta r_2}\right)\sin\lambda z\right]dz + \frac{\varphi_2(i\rho\beta r_2)}{\varphi_2(i\beta r_2)}\cos\lambda\beta r_2;$$

$$\varphi_2(i\rho z) = H_0^{(1)}(i\rho z)J_1(inz) - J_0(i\rho z)H_1^{(1)}(inz) = \frac{2}{\pi}\left[k_0(\rho z)I_1(nz) - K_1(nz)I_0(\rho z)\right],$$

is a solution of equation (1) satisfying conditions (2), (3), and (7).

Considering A as a function of the parameter β, we construct the integral

$$T(\rho;\lambda) = \int_{-\infty}^{0} A(\beta)\varphi(\rho,\lambda,\beta)\,d\beta, \tag{14}$$

in which $A(\beta)$ is determined by equality (6).

The function (14) satisfies equation (1) and the boundary conditions (2), (3), and (4).

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References

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Note: Figure translations are in progress. See original paper for figures.