MATHEMATICS

1961

SovietRxiv

Abstract

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MATHEMATICS

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ON MAJORANT EQUATIONS

(Presented by Academician V. I. Smirnov, 4 III 1961)

In the monographs (^{1,2}) L. V. Kantorovich, introducing the notion of a majorant of an operator, investigated an operator equation of the form

$$x = U(x)$$

in B_R- and B-spaces, establishing a number of important results concerning its solution (in (^{2}) the operator $U(x)$ is assumed to be continuously differentiable).

In the present work the notion of a strict majorant of an operator is introduced, making it possible to formulate a new criterion for the existence of a solution for an operator equation of the form

$$y = \Omega(y)$$

in complete metric spaces, and also to establish for it a number of quantitative results; here no conditions more restrictive than those imposed in Banach' s principle are placed on the operator Ω.

It is shown that for one class of metric spaces (containing the class of B-spaces) it is possible effectively to construct, in a definite sense, the best of the strict majorants. The results obtained in this case in principle cannot be achieved by applying Banach' s principle.

We shall use the following notation: Y is a complete metric space; Ω is an operator defined on Y, with $\Omega(Y) \subset Y$; $S_R(y_0)$ is the set of elements of Y satisfying the condition $\rho(y, y_0) < R$ $(R > 0)$; $F(x)$ is a real function of a real argument, defined on $[0, R)$.

Definition 1. We shall call a function $F(x)$ a **strict majorant** of the operator $\Omega(y)$ in $S_R(y_0)$ if: 1) it is increasing and convex; 2) $\rho[y_0, \Omega(y_0)] \le F(0)$; 3) from $y_1 \in S_R(y_0)$, $y_2 \in S_R(y_0)$, and $y_1 \neq y_2$ it follows that

$$\frac{\rho[\Omega(y_2), \Omega(y_1)]}{\rho(y_2, y_1)} \le \frac{F[\rho(y_0, y_2)] - F[\rho(y_0, y_1)]}{\rho(y_0, y_2) - \rho(y_0, y_1)} \;{}^{*}.$$

It can be shown that, in order that the operator $\Omega(y)$ have in $S_R(y_0)$ a strict majorant, it is necessary and sufficient that in $S_R(y_0)$ it satisfy a Lipschitz condition.

Definition 2. Let $M(y_0, \Omega)$ be the set of all positive numbers r having the property that in $S_r(y_0)$ there lies only one solution of the equation

$$y = \Omega(y). \tag{1}$$

Obviously, $M(y_0, \Omega)$ is an interval (possibly empty). We shall call the equation

$$x = F(x) \tag{2}$$

y_0-strictly majorant for equation (1), if $M(0, F) \subset M(y_0, \Omega)$.*

It is clear from the definition that if equation (2) is y_0-strictly majorant for equation (1) and $M(0, F)$ is nonempty, i.e. $M(0, F) = (a, b)$, where $a < b$, then: 1) a is the smallest root of equation (2), and b is either the second largest root (if $b < R$), or the right endpoint of the interval on which the function $F(x)$ is defined; 2) equation (1) has a solution y satisfying the inequality $\rho(y_0, y) \leq a$; 3) if equation (1) has solutions different from y, then all of them lie outside $S_b(y_0)$.

Theorem 1. Let $F(x)$ be a strict majorant of the operator $\Omega(y)$ in $S_R(y_0)$. Then the equation $x = F(x)$ is y_0-strictly majorant for the equation $y = \Omega(y)$.

It is clear from the theorem that knowledge of a strict majorant of the operator $\Omega(y)$ gives important information about the solution of the equation $y = \Omega(y)$. We shall give one property of strict majorants.

Let $\bar{R}$ be the greatest of the radii of the spheres $S_R(y_0)$ in which the operator $\Omega(y)$ satisfies the Lipschitz condition. Introduce the functions $\hat{f}(x)$ and $\widetilde{F}(x)$, defined on $[0, \bar{R})$:

$$\hat{f}(x) = \sup_{\substack{\rho(y_2, y_0) < x \\ \rho(y_1, y_0) < x \\ y_1 \neq y_2}} \frac{\rho[\Omega(y_2), \Omega(y_1)]}{\rho(y_2, y_1)}, \qquad \widetilde{F}(x) = \rho[y_0, \Omega(y_0)] + \int_0^x \hat{f}(t)\, dt.$$

The following simple statement holds.

Proposition. If $F(x)$ is a strict majorant of the operator $\Omega(y)$ on $S_R(y_0)$, then on the interval $[0, R)$ the inequality

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* If $\rho(y_0, y_2) = \rho(y_0, y_1) = x_0$, then the right-hand side of the inequality should be understood as the upper derivative of $F(x)$ at the point x_0.

$$F(x) \geq \widetilde{F}(x)$$

holds.

In this connection it is important to study the question: is the function $\widetilde{F}(x)$ always a strict majorant of the operator $\Omega(y)$? We do not have an answer to this question in the case when Y is an arbitrary complete metric space. It is possible, however, to indicate a class of spaces in which this question can be answered affirmatively. The results obtained in this direction are set forth below.

Definition. We shall call a complete metric space Y an **A-space** (after A. D. Aleksandrov, who first considered such objects in the monograph (^{3})) if it has the following properties:

1. Any two points of this space y_1 and y_2 can be joined by a curve** $L(y_1, y_2)$, all points of which also belong to Y, and whose length is equal to $\rho(y_1, y_2)$ (there exists a shortest path).
2. From $y \in L(y_1, y_2)$ and $y_0 \in Y$ it follows that

$$\rho(y_0, y) - \rho(y_0, y_1) \leq \frac{\rho(y, y_1)}{\rho(y_2, y_1)} [\, \rho(y_2, y_0) - \rho(y_1, y_0) \,],$$

i.e. the metric $\rho(y_1, y_2)$ is convex.***

* If the smallest root x_1 of equation (2) is multiple (i.e. x_1 is a minimum point of the function $F(x) - x$), then in defining $M(0, F)$ we count it as two equal roots. In this case, obviously, $M(0, F)$ is empty.

** We understand here the terms "curve" and "length of a curve" in the sense of A. D. Aleksandrov (^{3}).

*** We give a different definition of convexity of the metric than in (^{3}).

It is easy to see that B-spaces are a special case of A-spaces.

Theorem 2. If Y is an A-space, then $\widetilde{F}(x)$ is a strict majorant of the operator $\Omega(y)$.

In cases where $f(x)$ is not constant, $\widetilde{F}(x)$ is a strict majorant of the operator $\Omega(y)$, $M(0, \widetilde{F})$ is nonempty, and $b < R$, the estimates obtained for the solution of equation (1) and for the domain of uniqueness cannot be obtained with the aid of Banach' s principle (this is seen, for example, from the fact that $M(0, \widetilde{F})$ contains points where $f(x) > 1$).

In conclusion we note that Theorems 1 and 2, for the sake of brevity, have been formulated here in a somewhat weakened version.

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Received
23 II 1961

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Note: Figure translations are in progress. See original paper for figures.

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