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**Abstract**

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# Integral Formulas for Siegel Domains of the Second Kind

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In the present note we consider the Bergman and Cauchy-Szegő integral kernels for Siegel domains of the second kind ([1]). The explicit form of the kernels is found for affinely homogeneous domains associated with self-adjoint cones and, in particular, for all symmetric domains. Integral formulas for classical domains in another realization were obtained earlier by Hua Loo-keng ([2]), and for some homogeneous Siegel domains of the first kind by Bochner ([3]).

Let $\mathfrak{D}$ be a Siegel domain of the second kind, i.e. the set of points

$$\zeta = (z, u) \in C^{n+m}, \quad z \in C^n, \quad u \in C^m,$$

such that

$$\operatorname{Im} z - F(u, u) \in V,$$

where $V$ is a convex cone in $R^n$ containing no straight lines, and $F(u, v)$ is a $V$-Hermitian form. The Bergman kernel of the domain $\mathfrak{D}$ is the function $B(\eta, \zeta)$ on $\mathfrak{D} \times \mathfrak{D}$, analytic in $\eta$ and antianalytic in $\zeta$, for which

$$\int_{\mathfrak{D}} B(\eta, \zeta) f(\zeta)\, d\zeta\, d\bar{\zeta} = f(\eta),$$

where $f$ is an arbitrary bounded analytic function with integrable square of the modulus in $\mathfrak{D}$, continuous in $\overline{\mathfrak{D}}$. For simplicity we shall give the definition of the Cauchy-Szegő kernel only as applied to Siegel domains. The Cauchy-Szegő kernel is a function $S(\eta, \zeta)$ on $\mathfrak{D} \times \overline{\mathfrak{D}}$, possessing the same analyticity properties as $B$, and such that

$$\int_{\Omega_{\mathfrak{D}}} S(\eta, \zeta) f(\zeta)\, dx\, du\, d\bar{u} = f(\eta),$$

where

$$\Omega_{\mathfrak{D}} = \{\zeta = (z, u) : \operatorname{Im} z = F(u, u)\}$$

is the skeleton of the boundary of $\mathfrak{D}$, $x = \operatorname{Re} z$, and $f$ satisfies the same conditions as above.

Analytic automorphisms of the domain lead to functional equations for the Bergman kernel. This is expressed by the fact that $B(\zeta, \zeta)$ is the density of an invariant measure in $\mathfrak{D}$. In the case of a homogeneous domain, the resulting equations together with the analyticity condition are sufficient to determine $B$ up to a constant factor. The equations take an especially simple form for linear automorphisms. The use of automorphisms of the domain in finding the Cauchy-Szegő kernel causes certain difficulties. However, in the case of linear automorphisms these difficulties are absent, and one obtains very simple relations also for the kernel $S$. Therefore it is convenient to carry out the computation of the kernels for homogeneous domains by realizing them as Siegel domains, since in this case there is a transitive group of linear transformations.

We now proceed to the direct computation of the kernels. First of all, note that it is sufficient to determine $B(\eta,\zeta)$ and $S(\eta,\zeta)$ for $\eta=\zeta$, after which the kernels are uniquely found by means of the analyticity condition. If $g$ is a linear transformation of $\mathfrak{D}$, then

$$B(\eta,\zeta)=B(g\eta,g\zeta)j^2(g),$$

where $j(g)$ is the Jacobian of $g$ (by linearity of $g$ it does not depend on $\zeta$);

$$S(\eta,\zeta)=S(g\eta,g\zeta)\tilde{j}(g),$$

where

$$\tilde{j}(g)=d(gx)\,d(gu)\,d(\overline{gu})/dx\,du\,d\bar{u}.$$

Consider, in particular, the nilpotent group of automorphisms

$$N(\mathfrak{D}):(z,u)\to(z+a+2iF(u,c)+iF(c,c),\,u+c),$$

where $a\in R^n$, $c\in C^m$. Then $\tilde{j}=j=1$, and as a result it is enough to determine $B$ and $S$ for

$$\zeta=\eta=(iy,0),\quad y\in V.$$

Let, for these values of the argument,

$$B(\zeta,\zeta)=b(y),\qquad S(\zeta,\zeta)=s(y).$$

Then for

in order to obtain $B$ and $S$, in $b$ and $s$ one must replace the argument $y$ by

$$\rho(\eta,\xi)=(w-\bar{z})/2i-F(v,u),$$

where $\eta=(w,v)$, $\xi=(z,u)$.

We now restrict ourselves to the case of an affinely homogeneous Siegel domain. In this case the cone $V$ is also affinely homogeneous, and there exists a simply transitive group $K(V)$ of linear transformations of $V$ that extend to linear transformations of $\mathfrak{D}$. Let $(\lambda,\mu)$ be some scalar product in $R^n$, and let $e$ be a fixed point in $V$ such that $\sigma(\lambda)=(e,\lambda)>0$ for all $\lambda\in V$. Denote by $T_\lambda$

the mapping from $K(V)$ that carries $e$ to $\lambda$, by $\widetilde{T}_\lambda$ the extension of $T_\lambda$ to $C^m$ $(T_\lambda F(u,u) = F(\widetilde{T}_\lambda u, \widetilde{T}_\lambda u))$, and by $g_\lambda$ the resulting transformation of $\mathfrak{D}$. Then

$$b(y) = c_b j_z^{-2}(g_y) = c_b j_z^{-2}(y) j_u^{-2}(y); \qquad s(y) = c_s \tilde{j}^{-1}(g_y) = c_s j_z^{-1}(y) j_u^{-2}(y),$$

where

$$j_z(y) = d(T_y z)/dz, \qquad j_u(y) = d(\widetilde{T}_{yu})/du.$$

It remains for us to determine the numerical factors $c_b$ and $c_s$. The following integral representations for the kernels hold:

$$\int_V j_u^2(\lambda) \exp(i\sigma(\rho(g_\lambda \eta, g_\lambda \xi)))\, d\lambda_1 \cdots d\lambda_n = \frac{\Gamma_s S(\eta,\xi)}{c_s}, \tag{1}$$

$$\int_V j_u^2(\lambda) j_z(\lambda) \exp(i\sigma(\rho(g_\lambda \eta, g_\lambda \xi)))\, d\lambda_1 \cdots d\lambda_n = \frac{\Gamma_b B(\eta,\xi)}{c_b}. \tag{2}$$

These representations are obtained analogously to the generalized Siegel integral [6], into which they pass for $\eta = \xi$, $u = 0$. In view of this, the computation of $\Gamma_s$ and $\Gamma_b$ reduces to the computation of Siegel integrals. Applying the operators (1) and (2) to the functions $\exp(i\sigma(T_\lambda z/2))$, $\lambda \in V$, we obtain that the expressions (1) and (2) are respectively equal to $\alpha\beta S(\eta,\xi)$ and $\alpha\beta\gamma B(\eta,\xi)$, where

$$\int_{R^n} \exp\left(i\frac{\sigma(T_\lambda x - T_\mu x')}{2}\right) dx = \alpha\delta(\lambda - \mu), \qquad \beta = \int_{C^m} \exp(-\sigma(F(u,u)))\, du\, d\bar{u},$$

$$\gamma = \int_V \exp(-\sigma(y))\, dy.$$

The last integral is also a special case of the Siegel integral. Thus,

$$c_s = \Gamma_s/\alpha\beta, \qquad c_b = \Gamma_b/\alpha\beta\gamma.$$

Let us now consider the case when the cone $V$ is self-dual with respect to the chosen bilinear form $(\lambda,\mu)$. There exist four series of (classical) self-dual cones and one exceptional self-dual cone [4,5]. In the course of the computations the explicit form of the Siegel integral is used, and examples of Siegel domains connected with self-dual cones are also considered. The results relevant here for the classical cones are contained essentially in [6,7], while for the exceptional cone they are new. We shall investigate each of the types of cones.

1. Let $V$ be the cone of symmetric positive-definite matrices $\lambda$ of order $p$. Denote by $\Delta_i(\lambda)$ the principal minor formed by the first $i$ rows and columns. Put

$$(\lambda,\mu) = sp(\lambda\mu); \qquad e = E.$$

In this case the space $C^m$ decomposes into the direct sum of subspaces

$$C, \dots, C^{s_p}, \qquad s_1 \geq \dots \geq s_p,$$

and in $C^{s_i}$ is concentrated the component $F_{ii}(u,v)$ of the vector-function $F(u,v)$ (one may realize $C^m$ as a collection of rectangular matrices of order $p \times s_1$, in which in the $i$-th row only the last $s_i$ elements can be different from zero), putting

$$F(u,v) = \frac{1}{2}(uv^* + \overline{vu'}).$$

We have

$$j_z(y) = (\Delta_p(y))^{(p+1)/2}; \qquad j_u^2(y) = \prod_{i=1}^{p} (\Delta_i(y))^{s_i - s_{i+1}};$$

$$\Gamma_s = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(s_i + \frac{p-i}{2} + 1\right); \qquad \Gamma_b = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(s_i + p - \frac{i-3}{2}\right);$$

$$\alpha = 2^{p(p+3)/2} \pi^{p(p+1)/2}; \qquad \beta = \pi^m, \qquad m = \sum_{i=1}^{p} s_i; \qquad \gamma = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{p-i}{2} + 1\right).$$

2. For the cone of Hermitian positive-definite matrices of order $p$, just as in the preceding case, $(\lambda,\mu)$, $e$, $\Delta_i(\lambda)$ are defined, and the space $C^m$ is decomposed into a direct sum of subspaces $C^{s_i}$. Examples of $V$-Hermitian forms associated with homogeneous Siegel domains may be furnished by forms defined on the space of pairs of matrices $u^{(1)}, u^{(2)}$ of orders $p \times s_1^{(1)}$ and $p \times s_1^{(2)}$, respectively, for which in the $i$-th row only the last $s_i^{(1)}$ ($s_i^{(2)}$) elements are nonzero; $s_1^{(1)} \geq \dots \geq s_p^{(1)}$, $s_1^{(2)} \geq \dots \geq s_p^{(2)}$; $s_i = s_i^{(1)} + s_i^{(2)}$. The vector-function $F(u,v)$ is given by the formula

$$F(u,v) = u^{(1)} v^{(1)*} + \bar{v}^{(2)} u^{(2)\prime}.$$

In this way not all Siegel domains associated with this cone can be realized. One may, for example, also replace the space spanned by the first $s_i^{(1)} - s_{i+1}^{(1)}$ elements of the $i$-th row of the matrix $u^{(1)}$ and the first $s_i^{(2)} - s_{i+1}^{(2)}$ elements of the $i$-th row of the matrix $u^{(2)}$ by any of its subspaces. In the case under consideration we have:

$$j_z(y) = (\Delta_p(y))^p; \qquad j_u^2(y) = \prod_{i=1}^{p} (\Delta_i(y))^{s_i - s_{i+1}}; \qquad \Gamma_s = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(s_i$$

$$-i+p+1); \qquad \Gamma_b = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(s_i - i + 2p + 1); \qquad \alpha = 2^{p(p+1)/2} \pi^{p^2}; \quad \beta = \pi^m,$$

$$m = \sum_{i=1}^{p} s_i; \qquad \gamma = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(p - i + 1).$$

3. Let now $V$ be the cone of Hermitian positive-definite quaternionic matrices of order $p$, or, equivalently, of Hermitian positive-definite matrices $\lambda$ of order $2p$ for which

$$\lambda J = J\bar{\lambda}, \qquad J = \begin{pmatrix} j\cdot & 0 \\ 0 & j\cdot \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Set

$$(\lambda, \mu) = \tfrac{1}{2}\,\mathrm{sp}(\lambda\mu);$$

by $\Delta_i$ we shall denote the principal minor of order $2i$. The space $C^m$ decomposes into a direct sum of $p$ subspaces $C^{s_i}$ ($s_i \geq s_j$ for $i < j$), on which, respectively, the components $F_{2i-1,\,2i-1} = F_{2i,\,2i}$ of the form $F$ are concentrated. For example, as $C^m$ one may take the space of rectangular matrices of order $2p \times t_1$, in which in the $i$-th row only the last $t_i$ elements are nonzero, $t_{i+1} \leq t_i$; $s_i = t_{2i-1} + t_{2i}$, and set

$$F(u, v) = \tfrac{1}{2}(uv^* + \bar{J}v(Ju)').$$

One may also replace each space spanned by the first $(t_{2i-1} - t_{2i+1})$ elements in the $(2i-1)$-st and $(2i)$-th rows by any of its subspaces. Finally:

$$j_z(y) = (\Delta_p(y))^{p-\frac{1}{2}}; \qquad j_u^2(y) = \prod_{i=1}^{p} (\Delta_i(y))^{(s_i - s_{i+1})/2}$$

$$\Gamma_s = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(s_i - 2i + 2p + 1); \qquad \Gamma_b = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(s_i - 2i + 4p);$$

$$\alpha = 2^{2p^2} \pi^{p(2p-1)}; \qquad \beta = \pi^m, \quad m = \sum_{i=1}^{p} s_p; \qquad \gamma = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(2p - 2i + 1).$$

4. We proceed to consider the cone in $R^{n+2}$ consisting of points for which

$$|\lambda| = \lambda_1\lambda_2 - \lambda_3^2 - \dots - \lambda_{n+1}^2 > 0; \qquad \lambda_1 > 0.$$

Put

$$(\lambda, \mu) = \lambda_1\mu_1 + \lambda_2\mu_2 + 2\sum_{3}^{n+2} \lambda_k\mu_k; \qquad e = (1, 1, 0, \dots, 0).$$

The space $C^m$ is decomposed into the direct sum of subspaces $C^{s_1}$ and $C^{s_2}$, on one of which the component $F_1(u, v)$ is concentrated, and on the other $F_2(u, v)$; $s_1 \geq s_2$. For example, one may consider the space consisting of triples $(u_0, u_1, u_2)$, where $u_0 \in C^t$, $u_1 \in C^N$, $s_1 = t + N$, and $u_2 \in C^{s_2}$ ($C^{s_2}$ is some fixed subspace in $C^N$). The vector-function $F(u, v)$ is given by the formulas:

$$F_1(u, v) = (u_0, v_0) + (u_1, v_1); \qquad F_2(u, v) = (u_2, v_2); \qquad F_{k+2}(u, v) = \tfrac{1}{2}((u_1, T_{kv}2) +$$

$$+(T_k u_2, v_1)),$$

where $(u, v)$ is the ordinary scalar product, and $T_i$ $(i = 1, \dots, n)$ are unitary matrices of order $N$ satisfying the condition:

$$T_{kT} m^* + T_{mT} k^* = 0 \quad \text{for } m \neq k.$$

We have:

$$j_z(y) = |y|^{n/2+1}; \quad j_u^2(y) = |y|^{s_1} y_2^{-s_1+s_2};$$
$$\Gamma_s = \pi^{n/2}\Gamma(s_1 + n/2 + 1)\Gamma(s_2 + 1);$$
$$\Gamma_b = \pi^{n/2}\Gamma(s_1 + n - 2)\Gamma(s_2 + n/2 + 2);$$
$$\alpha = 2^{n+4}\pi^{n+2}; \quad \beta = \pi^m, \quad m = s + N + t; \quad \gamma = \pi^{n/2}\Gamma(n/2 + 1).$$

5. Finally, let us consider the special cone $V$ of Hermitian positive-definite octavic matrices of the third order $\lambda = \|\lambda_{ik}\|^*$. It consists of Hermitian octavic matrices representable in the form $\lambda = T_\lambda^* T_\lambda$, where $T_\lambda$ is an upper triangular octavic matrix of the 3rd order with positive real entries on the main diagonal. The matrices $T_\lambda$ form a group which acts simply transitively on $V$ by the formula

$$T_\mu \lambda = (T_\lambda T_\mu)^* (T_\lambda T_\mu).$$

Let

$$T_\lambda = \|t_{ij}(\lambda)\|; \quad \Delta_1(\lambda) = \lambda_{11} = t_{11}^2(\lambda),$$
$$\Delta_2(\lambda) = t_{11}^2(\lambda) t_{22}^2(\lambda), \quad \Delta_3(\lambda) = t_{11}^2(\lambda) t_{22}^2(\lambda) t_{33}^2(\lambda).$$

The invariant volume element in $V$ has the form

$$dv = (\Delta_3(\lambda))^{-9}\, d\lambda,$$

where $d\lambda$ is the Euclidean volume element;

$$(\lambda, \mu) = \mathrm{sp}(\lambda\mu) = \sum_{i=1}^{3} \lambda_{ii}\mu_{ii} + 2\,\mathrm{Re}\sum_{i<j} \lambda'_{ij}\mu_{ij}.$$

Let

$$e_{\alpha_1,\alpha_2,\alpha_3} = \Delta_1^{\alpha_1}\Delta_2^{\alpha_2}\Delta_3^{\alpha_3}.$$

The formula holds

$$\int_V e^{-(\lambda,\mu)} e_{\alpha_1,\alpha_2,\alpha_3}(\mu)\, dv(\mu) = G_{\alpha_1,\alpha_2,\alpha_3} e_{\alpha_1,\alpha_2,\alpha_3}(\lambda^{-1}),$$

where

$$G_{\alpha_1,\alpha_2,\alpha_3} = \pi^{12}\Gamma(\alpha_1 + \alpha_2 + \alpha_3)\Gamma(\alpha_2 + \alpha_3 - 4)\Gamma(\alpha_3 - 8).$$

The space $C^m$ decomposes into the direct sum of three subspaces $C^{s_1}, C^{s_2}, C^{s_3}$, $s_1 \geq s_2 \geq s_3$, and $F_{ii}(u,v)$ is concentrated on $C^{s_i}$. Let us give examples. We shall call a complex octavic linear combination with complex coefficients of the basis elements of the octavic algebra

$$a = \sum_{k=0}^{7} a_{ke}k, \quad a_k \in C.$$

Put:

$$a' = a_0 e_0 - \sum_{k=1}^{7} a_{ke}k, \quad \bar{a} = \sum_{k=0}^{7} \bar{a}_{ke}k, \quad a^* = (\bar{a})'.$$

As the space $C^m$ we take the space of rectangular matrices of three rows whose elements are complex octaves, with all elements in the $i$-th row vanishing except the last $t_i$, $t_1 \geq t_2 \geq t_3$, $8t_i = s_i$. The elements of the group $T_\lambda$ act in this space by multiplication on the left. Further,

$$F_{ij}(u,v) = \sum_{k=1}^{t_1} u_{ik}v^*_{kj} + \bar{v}_{ik}u'_{kj}.$$

Moreover, as in the preceding examples, one may replace the space generated by the first $s_i - s_{i+1}$ elements of the $i$-th row by an arbitrary subspace. Finally we obtain:

$$j_z(y) = (\Delta_3(y))^9; \quad j_u^2(y) = \prod_{i=1}^{3} (\Delta_i(y))^{s_i - s_{i+1}}$$

$$\Gamma_s = \pi^{12}\Gamma(s_1+9)\Gamma(s_2+5)\Gamma(s_3+1);$$

$$\Gamma_b = \pi^{12}\Gamma(s_1+18)\Gamma(s_2+14)\Gamma(s_3+10);$$

$$\alpha = 2^{30}\pi^{27}; \quad \beta = \pi^{(s_1+s_2+s_3)}; \quad \gamma = \pi^{12}\Gamma(9)\Gamma(5).$$

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* For the definition of octaves (Cayley numbers), see, for example, in (8).

*Note: Figure translations are in progress. See original paper for figures.*