ON THE CONSERVATION OF VORTICES AND CURRENTS IN MAGNETOHYDRODYNAMICS

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Abstract

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ON THE CONSERVATION OF VORTICES AND CURRENTS IN MAGNETOHYDRODYNAMICS

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In classical hydrodynamics Thomson' s theorem is valid, asserting that the circulation of velocity along any closed material fluid contour remains constant throughout the entire time of motion, or that vortices move together with the fluid particles. It is easy to verify that in magnetohydrodynamics this theorem is not satisfied. However, here there is a valid assertion which, to a certain degree, plays the same role as Thomson' s theorem. The purpose of the present work is to prove the corresponding theorem.

Let us consider the flow of an incompressible fluid with infinite electrical conductivity. We shall regard the flow as weakly perturbed, i.e., we assume that $\mathbf{V} - \mathbf{V}_0$ and $\mathbf{H} - \mathbf{H}_0$ and their derivatives are small, so that squares of these quantities may be neglected. Here $\mathbf{V}$ and $\mathbf{H}$ are, respectively, the velocity and magnetic-field vectors; quantities referring to the basic unperturbed flow are denoted by the subscript zero.

Instead of the field $\mathbf{H}$, we introduce the corresponding Alfvén velocity $\mathbf{A} = \mathbf{H}/\sqrt{4\pi\rho}$, where ρ is the fluid density.

Consider an arbitrary contour moving either with velocity $\mathbf{V}_0 + \mathbf{A}_0$, or with velocity $\mathbf{V}_0 - \mathbf{A}_0$. For this contour one can formulate the following theorem:

Theorem. *In a weakly perturbed incompressible fluid with infinite electrical conductivity, the circulation of the vector* $\mathbf{V} \pm \mathbf{A}$ *along an arbitrary contour moving with velocity* $\mathbf{V}_0 \mp \mathbf{A}_0$ *is constant.*

Consider a contour moving with velocity $\mathbf{V}_0 - \mathbf{A}_0$. Since the contour is nondeformable, we have

$$\frac{d}{dt}\oint(\mathbf{V} + \mathbf{A},\, d\mathbf{r}) = \oint\left(\frac{d}{dt}(\mathbf{V} + \mathbf{A}),\, d\mathbf{r}\right) \quad \left(\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_0 - \mathbf{A}_0, \nabla)\right). \quad (1)$$

The equations of motion and induction of magnetohydrodynamics, after linearization and introduction of the Alfvén velocity in place of the field, may be written in the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}_0, \nabla)\mathbf{V} = -\operatorname{grad}\left(\frac{p}{\rho} + \frac{A^2}{2}\right) + (\mathbf{A}_0, \nabla)\mathbf{A}, \tag{2}$$

$$\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{V}_0, \nabla)\mathbf{A} = (\mathbf{A}_0, \nabla)\mathbf{V}, \tag{3}$$

where p is the pressure.

Adding (2) and (3), we obtain

$$\frac{\partial(\mathbf{V} + \mathbf{A})}{\partial t} + (\mathbf{V}_0, \nabla)(\mathbf{V} + \mathbf{A}) = (\mathbf{A}_0, \nabla)(\mathbf{V} + \mathbf{A}) - \operatorname{grad}\left(\frac{p}{\rho} + \frac{A^2}{2}\right).$$

or

$$\frac{d(\mathbf{V} + \mathbf{A})}{dt} = -\operatorname{grad}\left(\frac{p}{\rho} + \frac{A^2}{2}\right). \tag{4}$$

Substituting (4) into (1), we verify the validity of the theorem for a contour moving with velocity $\mathbf{V}_0 - \mathbf{A}_0$.

To prove the theorem for a contour moving with velocity $\mathbf{V}_0 + \mathbf{A}_0$, subtracting (3) from (2), we obtain

$$\frac{d(\mathbf{V} - \mathbf{A})}{dt} = -\operatorname{grad}\left(\frac{p}{\rho} + \frac{A^2}{2}\right), \quad \left(\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_0 + \mathbf{A}_0, \nabla)\right). \tag{5}$$

Composing an expression analogous to (1) for the circulation of the vector $(\mathbf{V} - \mathbf{A})$, we verify the validity of this part of the theorem as well.

The result obtained can be represented in a somewhat different form. Applying the operation rot to equations (4) and (5), we obtain

$$\frac{\partial \vec{\omega}_1}{\partial t} + (\mathbf{V}_0 - \mathbf{A}_0, \nabla)\vec{\omega}_1 = 0, \qquad \frac{\partial \vec{\omega}_2}{\partial t} + (\mathbf{V}_0 + \mathbf{A}_0, \nabla)\vec{\omega}_2 = 0, \tag{6}$$

where $\vec{\omega}_1 = \operatorname{rot}(\mathbf{V} + \mathbf{A})$ and $\vec{\omega}_2 = \operatorname{rot}(\mathbf{V} - \mathbf{A})$.

Thus, $\operatorname{rot}(\mathbf{V} + \mathbf{A})$ is displaced relative to the fluid with velocity $-\mathbf{A}_0$, while $\operatorname{rot}(\mathbf{V} - \mathbf{A})$ is displaced with velocity $\mathbf{A}_0$; that is, unlike in ordinary hydromechanics, there are two possible directions of displacement of vortices, and the vortices are not frozen into the fluid.

The result obtained makes it possible to clarify the mechanism of formation of wakes and vortex trails that arise near bodies moving in a fluid.

Fig. 1

Figure 1: Fig. 1

Consider the motion of a body in an incompressible viscous fluid with finite electrical conductivity. At a sufficiently large distance from the body the flow will always be weakly disturbed, and the equations can be linearized (Oseen approximation).

Obviously, according to what has been proved, the vortices arising near the body will move in the directions $(\mathbf{V}_0 - \mathbf{A}_0)$ and $(\mathbf{V}_0 + \mathbf{A}_0)$, and the presence of viscosity and electrical conductivity will lead to a blurring of the resulting pattern (Fig. 1).

Fig. 1

For the particular case of flow past a cylinder in a magnetic field parallel to the velocity of the incident stream, the Oseen problem has been considered in work (1). In accordance with what has been said above, for finite electrical conductivity there are two vortex wakes, one of which, for $\mathbf{A}_0 > \mathbf{V}_0$, goes upstream.

Let us note that for not small disturbances the circulation of the vectors $\mathbf{V} \pm \mathbf{A}$ is conserved, respectively, along contours moving with velocity $\mathbf{V} \mp \mathbf{A}$, if throughout the flow $\mathbf{V} \parallel \mathbf{H}$, or if the flow in the spaces of the hodograph of velocity and field degenerates into a line.

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CITED LITERATURE

1. H. Yosinobu, *J. Phys. Soc. Japan*, **15**, No. 1 (1960).

Note: Figure translations are in progress. See original paper for figures.