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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON THE APPROXIMATION OF ANALYTIC  
FUNCTIONS IN THE MEAN OVER A DO-  
MAIN**

*(Presented by Academician V. I. Smirnov on 1 VIII 1960)*

Consider the class of functions  $H'_p$  ( $p \geq 1$ ), analytic in a bounded simply connected domain  $D$ , for which

$$\|f(z)\|_D = \left\{ \iint_D |f(z)|^p d\sigma_z \right\}^{1/p} < \infty, \quad z = x + iy, \quad d\sigma_z = dx dy. \quad (1)$$

For certain types of domains belonging to the Carathéodory class, we shall establish a connection between the integral modulus of continuity of a function  $f(z) \in H'_p$  in  $D$

$$\omega_p(\delta, f) = \sup_{|h| \leq \delta} \left\{ \iint_D |f(z+h) - f(z)|^p d\sigma_z \right\}^{1/p}, \quad (2)$$

$f(z) = 0$  outside  $D$ , and its best approximation in the mean  $\rho_n^{(p)}(f, D)$  by polynomials of degree  $n$ , which is defined as the exact lower bound of the numbers

$$\left\{ \iint_D |f(z) - Q_n(z)|^p d\sigma_z \right\}^{1/p}$$

over all possible polynomials  $Q_n(z)$  of degree  $n$ .

We shall use the inequality

$$\left\{ \iint_D d\sigma_z \left[ \int_a^b |F(z, u)| du \right]^p \right\}^{1/p} \leq \int_a^b du \left[ \iint_D |F(z, u)|^p d\sigma_z \right]^{1/p}, \quad (3)$$

which is valid when  $F(z, u)$  is defined for every pair of points  $(z, u)$ ,  $z \in D$ ,  $a \leq u \leq b$ , and the integrals exist. It can be obtained by the same method by which the generalized Minkowski inequality is established for a function of two real variables defined in a rectangle.

1°. We first present one result for functions of the class  $H'_p$  in the disk  $|z| < 1$ , which is an analogue of the well-known theorems of Hardy and Littlewood.

**Theorem 1.** In order that a function  $f(z)$  of the class  $H'_p$  in the disk  $|z| < 1$  satisfy the condition

$$\left\{ \iint_{|z|<1} |f(re^{i(\vartheta+\tau)}) - f(re^{i\vartheta})|^p r dr d\vartheta \right\}^{1/p} \leq A_1 |\tau|^\alpha, \quad (4)$$

$0 < \alpha \leq 1$ ,  $p \geq 1$ , it is necessary and sufficient that the inequality

$$\left\{ \int_0^{2\pi} \int_0^\rho |f'(re^{i\vartheta})|^p r dr d\vartheta \right\}^{1/p} \leq \frac{A_2}{(1-\rho)^{1-\alpha}}, \quad 0 < \rho < 1. \quad (5)$$

We note that here and in what follows the constants  $A$  with subscripts do not depend on  $\tau, \rho, n, \delta$ . Further in this item we consider a domain  $D$ , bounded by an analytic Jordan curve.

**Theorem 2.** If  $f^{(m)}(z) \in H'_p$  in  $D$  ( $p \geq 1$ ) and  $\omega_p(\delta, f^{(m)}) \leq A_3 \delta^\alpha$ ,  $0 < \alpha \leq 1$ , then  $\rho_n^{(p)}(f, D) \leq A_4 n^{-m-\alpha}$  for all natural  $n \geq 1$ ; here  $m$  is an integer  $\geq 0$ .

We indicate the course of the proof for the case  $m = 0$  and the domain  $D$  coinciding with the disk  $|z| < 1$ . From Cauchy's integral formula we have, for  $|z| \leq \rho$ ,  $\rho < 1$ ,

$$f'(z) = \frac{1}{2\pi i} \int_{L_\rho} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\zeta,$$

where  $L_\rho$  is the circle  $|\zeta - z| = \frac{1}{2}(1 - \rho)$ ; putting  $\zeta - z = u = \frac{1}{2}(1 - \rho)e^{i\vartheta}$ , we obtain, on the basis of (3),

$$\begin{aligned} \left\{ \iint_{|z| \leq \rho} |f'(z)|^p d\sigma_z \right\}^{1/p} &\leq \frac{1}{\pi} \left\{ \iint_{|z| \leq \rho} d\sigma_z \left[ \int_0^{2\pi} \frac{|f(z+u) - f(z)|}{1-\rho} d\vartheta \right]^p \right\}^{1/p} \leq \\ &\leq \frac{1}{\pi(1-\rho)} \int_0^{2\pi} d\vartheta \left\{ \iint_{|z| \leq \rho} |f(z+u) - f(z)|^p d\sigma_z \right\}^{1/p}. \end{aligned}$$

From the condition of the theorem there follows the estimate

$$\left\{ \iint_{|z| \leq \rho} |f'(z)|^p d\sigma_z \right\}^{1/p} \leq \frac{A_5}{(1-\rho)^{1-\alpha}};$$

hence, with the aid of Theorem 1, it follows that

$$\left\{ \iint_{|z| < 1} |f(re^{i(\vartheta+\tau)}) - f(re^{i\vartheta})|^p r dr d\vartheta \right\}^{1/p} \leq A_6 |\tau|^\alpha.$$

The proof is completed if we form the function

$$P_n(z) = \frac{3}{2\pi n(2n^2 + 1)} \int_{-\pi}^{\pi} f(re^{i\varphi}) \left[ \frac{\sin n \frac{\varphi-\vartheta}{2}}{\sin \frac{\varphi-\vartheta}{2}} \right]^4 d\varphi, \quad z = re^{i\vartheta},$$

which is a polynomial of degree  $\leq 2n-2$ , and use inequality (3) and the estimates from the proof of Jackson's theorem. We note that, for  $p > 1$  and the condition  $\omega_p(\delta, f) \leq A_7 \delta^\alpha$ ,  $0 < \alpha \leq 1$ , in the disk  $|z| < 1$  the estimate

$$\|f(z) - S_n(z)\|_{|z| < 1} < A_8 n^{-\alpha}, \quad S_n(z) = \sum_{k=0}^n a_k z^k$$

is valid, where  $S_n(z)$  is the partial sum of the Taylor series for  $f(z)$ .

The converse theorem on approximation in the mean is based on the following lemma.

**Lemma 1.** If  $P(z)$  is a polynomial of degree  $n$ , then

$$\|P'(z)\|_D \leq nH \|P(z)\|_D,$$

where the constant  $H$  depends only on the domain  $D$  and  $p \geq 1$ .

**Theorem 3.** If for all natural  $n \geq 1$  the inequality

$$\rho_n^{(p)}(f, D) \leq A_9 n^{-m-\alpha},$$

where  $m$  is an integer  $\geq 0$ ,  $0 < \alpha \leq 1$ , is satisfied, then  $f^{(m)}(z) \in H'_p$  in the domain  $D$  and the inequalities

$$\omega_p(\delta, f^{(m)}) \leq A_{10} \delta^\alpha \quad \text{for } 0 < \alpha < 1,$$

$$\omega_p(\delta, f^{(m)}) \leq A_{11} \delta(1 + |\ln \delta|) \quad \text{for } \alpha = 1$$

hold.

2°. One can establish a connection between the best approximation  $\rho_n^{(p)}(f, D)$  in the mean over the domain and an integral Lipschitz condition for  $f(z)$  on the boundary  $C$  of the domain  $D$ ; we again assume this boundary to be an analytic Jordan curve. We shall say that  $f(z) \in L_p(m, \alpha)$  on  $C$ , if  $f^{(m)}(z)$  belongs to the class  $E_p$  in the sense of V. I. Smirnov in  $D$  and

$$\left\{ \int_0^l |F(s + \tau) - F(s)|^p ds \right\}^{1/p} \leq A_{12} |\tau|^\alpha,$$

where  $F(s) = f^{(m)}[z(s)]$ ,  $z = z(s)$  is the equation of the line  $C$ ,  $s$  is the length of a variable arc on  $C$ ,  $l$  is the length of the curve  $C$ ,  $m$  is an integer  $\geq 0$ ,  $0 < \alpha \leq 1$ .

**Theorem 4.** *In order that  $f(z) \in L_p(m, \alpha)$ ,  $p \geq 1$ ,  $0 < \alpha < 1$ , it is necessary and sufficient that the inequality*

$$\rho_n^{(p)}(f, D) \leq A_{13} n^{-m-\alpha-1/p} \quad \text{for } n \geq 1$$

hold.

This theorem loses its meaning if  $\rho_n^{(p)}(f, D) \leq A_{14} n^{-\gamma}$ ,  $0 < \gamma \leq 1/p$ . In this case the following is valid.

**Theorem 5.** *The condition  $\rho_n^{(p)}(f, D) \leq A_{15} n^{1-\alpha-1/p}$ , where  $1 - 1/p < \alpha < 1$ ,  $p \geq 1$ , is necessary and sufficient in order that the function*

$$\int_a^z f(t) dt, \quad a \in D,$$

belong to the class  $L_p(0, \alpha)$  on  $C$ .

For the particular case  $p = 2$ , Theorems 4 and 5 were established in the work (2).

3°. Let now  $D$  denote an arbitrary bounded simply connected domain with simply connected complement. Using certain results and ideas of S. N. Mergelyan (3) in the corresponding questions of the theory of uniform approximation, we shall establish an upper estimate for  $\rho_n^{(p)}(f, D)$ .

For a function  $f(z) \in H'_p$  ( $p > 1$ ) in the domain  $D$ , form the function

$$\Phi_\delta(z) = \iint_{|\zeta-z|<1} f(\zeta) K_\delta(|\zeta-z|) d\xi d\eta, \quad \zeta = \xi + i\eta; \quad (6)$$

here  $\delta$  is a fixed number,  $0 < \delta < 1$ ,

$$K_\delta(r) = \frac{3}{\pi\delta^2} \left(1 - \frac{r}{\delta}\right) \quad \text{for } 0 < r \leq \delta, \quad K_\delta(r) = 0 \quad \text{for } r > \delta.$$

**Lemma 2.** *The function  $\Phi_\delta(z)$  is continuous in the whole plane,  $\Phi_\delta(z) = f(z)$  on the set  $D_\delta$  of points of  $D$  whose distance to the boundary  $C$  of the domain  $D$  is greater than  $\delta$ ,  $\|f(z) - \Phi_\delta(z)\|_{D'}$   $\leq 2\omega_p(\delta, f)$  for any finite domain  $D'$ ,  $\|\Psi_\delta(z)\|_{D'}$   $\leq 24\omega_p(\delta, f)/\delta$ , where  $\Psi_\delta(z) = (\partial U_\delta/\partial x - \partial V_\delta/\partial y) + i(\partial U_\delta/\partial y + \partial V_\delta/\partial x)$ ; here  $U_\delta(z) = \operatorname{Re} \Phi_\delta(z)$ ,  $V_\delta(z) = \operatorname{Im} \Phi_\delta(z)$ , and  $\omega_p(\delta, f)$  is defined according to (2) in the domain  $D$ .*

**Lemma 3.** *Let  $F(z)$  be an analytic function on the set  $D_\delta$ , continuous in the disk  $K$  with center at some point  $z_0 \in D$  and radius  $2d$ , where  $d$  is the diameter of the domain  $D$ . If the partial derivatives of  $u(z) = \operatorname{Re} F(z)$ ,  $v(z) = \operatorname{Im} F(z)$  with respect to  $x$  and  $y$  are summable to degree  $p > 1$  over the disk  $K$ , and  $\|\psi(z)\|_K \leq M$ , where  $\psi(z) = (u'_x - v'_y) + i(u'_y + v'_x)$ , then there exists a polynomial  $P(z)$  such that  $\|F(z) - P(z)\|_D \leq H_1 M \delta$ , where  $H_1$  is an absolute constant.*

Let us note that the proof of Lemma 3 is based on the formula

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi} \iint_\Delta \frac{\psi(\zeta)}{\zeta - z} d\xi d\eta,$$

which can be proved under the condition that  $\psi(z)$  is summable to degree  $p > 1$  over the domain  $D$ ,  $F(z)$  is continuous in the closure of  $\Delta$ , and the boundary  $\Gamma$  of the domain  $\Delta$ —

a smooth curve  $z = z(s)$  ( $s$  is the arc length on  $\Gamma$ ), for which

$$\int_0^c j(u) u^{-1} du < \infty, \tag{7}$$

where  $j(u)$  is the modulus of continuity of  $z'(s)$ . The proof of this formula is known<sup>4</sup> for  $p > 2$  for a domain with an arbitrary Jordan boundary.

Let the domain  $D_R$  be bounded by the line  $C_R$ , which is the image of the circle  $|w| = R > 1$  under the univalent mapping  $w = \Phi(z)$  of the complement of  $D$  onto  $|w| > 1$ ,  $\Phi(\infty) = \infty$ .

**Lemma 4.** *If  $S(z) \in H'_p$ , ( $p > 1$ ), in the domain  $D_R$  ( $R > 1$ ), and  $\|S(z)\|_{D_R} \leq M$ , then for  $n \geq 1$*

$$\rho_n^{(p)}(S, D) \ll BMn^9/R^n \tag{8}$$

for every  $R$ ,  $1 < R \leq R_0$ , where the constant  $B$  depends only on  $R_0$ .

Let  $d(\zeta, \mu)$  be the distance from the point  $\zeta$  on the boundary  $C$  of the domain  $D$  to the line  $C_R$  for  $R = 1 + \mu$ , and let  $d(\mu) = \sup_{\zeta \in C} d(\zeta, \mu)$  be the function

introduced in work<sup>3</sup>. With the aid of the last three lemmas the following theorem is established:

**Theorem 6.** If  $D$  is a bounded simply connected domain with simply connected complement and  $f(z) \in H'_p$ , ( $p > 1$ ), in the domain  $D$ , then for all natural  $n \geq 1$

$$\rho_n^{(p)}(f, D) \ll B_1 \omega_p \left[ d \left( \frac{10 \ln n}{n}, f \right), f \right], \quad (9)$$

where the constant  $B_1$  does not depend on  $n$ .

We note special cases of this theorem.

1. If the boundary of the domain  $D$  is a smooth curve with continuously rotating tangent, then for  $n \geq 1$  and any  $\varepsilon > 0$

$$\rho_n^{(p)}(f, D) \ll B_2 \omega_p \left( \frac{1}{n^{1-\varepsilon}}, f \right),$$

where the constant  $B_2$  depends on  $\varepsilon$ , but does not depend on  $n$ .

2. If for the smooth boundary of the domain  $D$  condition (7) is fulfilled, then

$$\rho_n^{(p)}(f, D) \ll B_3 \omega_p \left( \frac{\ln n}{n}, f \right)$$

for  $n \geq 1$ , where the constant  $B_3$  does not depend on  $n$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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