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Abstract

Full Text

THEORY OF ELASTICITY

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STABILITY OF ORTHOTROPIC PLATES OF STEPWISE VARIABLE STIFFNESS

(Presented by Academician Yu. N. Rabotnov, November 2, 1960)

§ 1. In this work we consider the problem of vibrations of an orthotropic square plate with a stepwise variation of its thickness. It is assumed that the plate is subjected to longitudinal compressive forces T, applied in the middle plane along the lines $x = a_\gamma$ (Fig. 1A).

Let one of the planes of elastic symmetry coincide with the middle plane of the plate, and let the other two planes be parallel to its sides. We place the coordinate axes ox, oy, situated in the middle plane of the plate, along the principal directions of elasticity and direct them along the freely supported edges of the plate. Suppose that the forces depend on one parameter T_0: $T_1 = \beta_1 T_0,\ T_2 = \beta_2 T_0, \dots,\ T_\nu = \beta_\nu T_0$, where β_γ are known numbers $(\gamma = 1, 2, \dots, \nu)$, and ν is the number of sections of the plate.

Fig. 1. Orthotropic plate of stepwise variable stiffness and arrangement of adjacent grid nodes

To solve the problem we apply a synthesis of the finite-difference method and the Bubnov–Galerkin method. We use a square grid and divide each side of the plate into n equal parts. The grid spacing is $\varepsilon = a/n$. The arrangement of adjacent nodes of the grid is shown in Fig. 1B.

The deflections of the plate u_0 for a grid of regular structure on each of its sections satisfy the system of difference equations

$$\begin{aligned} 2(3 + 3k_2 + 4k_3)u_0 - 4(k_2 + k_3)(u_2 + u_4) - 4(1 + k_3)(u_1 + u_3) \\ + 2k_3(u_5 + u_6 + u_7 + u_8) + k_2(u_{11} + u_{12}) + u_9 + u_{10} = \\ = \lambda \frac{P_\nu}{R_\nu}(2u_0 - u_1 - u_3) + Fu_0, \end{aligned} \tag{1}$$

Fig. 2. Graphs of values of the length coefficients

Figure 2: Fig. 2. Graphs of values of the length coefficients

where $P_\nu = (T_0 + T_1 + \cdots + T_\nu)/T_0$; $R_\nu = D_\nu/D_0$; $\lambda = a^2T_0/D_0n^2$; $F = \rho p^2a^4/D_0n^4$; $k_2 = D_2/D_0$; $k_3 = D_3/D_0$; $D_0 = E_0h^3/12(1-\mu_1\mu_2)-$ -

cylindrical rigidity of the first segment for the principal direction along the ox axis, D_γ is the same for the γ-th segment;
$D_2 = E_2h^3/12(1-\mu_1\mu_2)$; $D_3 = D_0\mu_2 + 2D_k$; $D_k = Gh^3/12$; p is the frequency of vibration; ρ is the mass of the plate per unit area of the middle surface. We shall take the distance between two adjacent grid lines as equal to $\varepsilon = a/6$.

Writing equalities (1) for all internal nodal points, taking into account the simple support of the plate along the contour, we obtain a system of linear homogeneous algebraic equations with respect to the unknown nodal deflections u_0.

The equation of the middle elastic surface satisfying the conditions of simple support at the boundaries of the plate will be taken in the form (1)

$$u_0 = u(\xi,\eta) = C\sin\pi\xi\sin\pi\eta, \tag{2}$$

where $\xi = x/a$ and $\eta = y/a$ are dimensionless coordinates. Carrying out the operations in accordance with (2), we obtain a homogeneous linear equation with respect to the parameters λ and F.

Fig. 2. Graphs of values of the length coefficients

The relation between the value of the natural vibration frequency p and the longitudinal compressive forces T for a square plate compressed by a stepped load along the lines $\xi = 0$, $\xi = 0.5$, and $\xi = 1$ has the form

$$p^2 = \frac{9.65\,(R_1+P_1)}{\rho a^2(R_1+1)}\left[\frac{19.34\,(1+k_2+2k_3)D_1}{a^2(R_1+P_1)} - T\right]. \tag{3}$$

Knowing the magnitude of the compressive forces T, we find the vibration frequency of the plate.

Taking $T = 0$ in equation (3), we obtain the fundamental natural vibration frequency

$$p = \frac{13.7}{a^2}\sqrt{\frac{(1+k_2+2k_3)D_1}{\rho(1+R_1)}}.$$

We shall represent the critical load in the form

$$T_{\text{cr}} = VD_\nu/a_{\text{red}}^2, \tag{4}$$

where V is a constant characterizing T_{cr} for a plate of constant thickness compressed along the contour in one direction; $a_{\text{red}} = \alpha a$ is the reduced side of the plate; α is a coefficient depending on the manner of loading and the boundary conditions of the plate. Thus, by virtue of formula (4), the problem reduces to finding the values of α for various ratios P_ν and R_ν.

Taking $\lambda P_\nu / R_\nu = V/\alpha^2$, we have

$$(T_0 + T_1)_{\text{cr}} = V D_1 / a_{\text{red}}^2. \tag{5}$$

Putting $p = 0$ in equation (3), we obtain the minimum value of the critical force

$$(T_0 + T_1)_{\text{cr}} = 9.67(1 + k_2 + 2k_3) D_1 / a_{\text{red}}^2. \tag{6}$$

Specifying various ratios P_ν and R_ν, we obtain values of α for any fixed case of loading of the plate. In Fig. 2 the values of the length coefficients α are given for the case $b/a = 0.5$.

By the same method one can study inhomogeneous plates for which $E = E(\xi, \eta)$.

§ 2. To take into account the influence of the rheological properties of the material on the value of the critical force of an orthotropic plate, we apply the Volterra–Rabotnov principle ([3]). Introduce three integral time operators:

$$\overline{E}_1 = E_{10}\,[1 - \delta_1 \mathcal{E}_\alpha^*(-\beta_1)], \qquad \overline{E}_2 = E_{20}\,[1 - \delta_2 \mathcal{E}_\alpha^*(-\beta_2)],$$
$$\overline{G} = G_0[1 - \delta_3 \mathcal{E}_\alpha^*(-\beta_3)], \tag{7}$$

where

$$\delta_1 = (E_{10} - E_{1\infty})/E_{10}; \qquad \delta_2 = (E_{20} - E_{2\infty})/E_{20}; \qquad \delta_3 = (G_0 - G_\infty)/G_0;$$
$$\beta_i = \delta_i \tau_i^{\alpha - 1} \; (i = 1, 2, 3), \quad 0 < \alpha < 1;$$

E_{10}, E_{20}, G_0 are the instantaneous moduli of elasticity and shear; $E_{1\infty}, E_{2\infty}, G_\infty$ are the relaxed moduli of elasticity; τ_1, τ_2, and τ_3 are the relaxation times for two types of longitudinal and one shear deformation.

$\mathcal{E}_\alpha^*(-\beta_i)$ is an integral operator acting on unity:

$$\mathcal{E}_\alpha^*(-\beta_i) \cdot 1 = \int_0^t \mathcal{E}_\alpha(-\beta_i s)\, ds \qquad (i = 1, 2, 3),$$

where $\mathcal{E}_\alpha(-\beta_i, s)$ is a special function of fractional order due to Yu. N. Rabotnov ([3]):

$$\mathcal{E}_\alpha(-\beta_i; s) = s^{-\alpha} \sum_{\nu=0}^{\infty} \frac{(-\beta)^\nu s^{\nu(1-\alpha)}}{\Gamma[(\nu+1)(1-\alpha)]},$$

Γ is the gamma function.

Following N. Kh. Arutyunyan ([4]), we assume that the parameters μ_1 and μ_2 do not vary with time.

Substituting (7) into equality (6) and taking into account the values of D_1, D_2, and D, we obtain:

$$T_{\text{cr}}(t) = T_{\text{cr}}(0) \left[1 - \sum_{i=1}^{3} a_i \mathcal{E}^*_\alpha(-\beta_i)\right],$$

where

$$a_1 = E_{10} h \delta_1 (1 + 2\mu_2)/T; \qquad a_2 = E_{20} h \delta_2 / T; \qquad a_3 = 4 G_0 h^3 \delta_3 (1 - \mu_1 \mu_2)/T;$$

$$T = E_{10} h (1 + 2\mu_2) + E_{20} h + 4 h^3 G_0 (1 - \mu_1 \mu_2).$$

Since, by its physical meaning, $T_{\text{cr}}(0)$ does not depend on time, one may use the relation established by M. I. Rozovskii ([5]):

$$\mathcal{E}^*_\alpha(-\beta) \cdot 1 = \frac{1}{\beta} [1 - E_{1-\alpha}(-\beta t^{1-\alpha})], \tag{8}$$

where $E_{1-\alpha}$ is the Mittag-Leffler function of order $\mu = 1 - \alpha$, defined by the series

$$E_\mu(-\xi) = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\xi^\nu}{\Gamma(\nu\mu + 1)}. \tag{9}$$

Taking (8) into account, we have

$$T_{\text{cr}}(t) = T_{\text{cr}}(0) \left\{1 - \sum_{i=1}^{3} \frac{a_i}{\beta_i} [1 - E_\mu(-\beta_i t^\mu)]\right\}. \tag{10}$$

Since $\lim_{\xi \to \infty} E_\mu(-\xi) = 0$, it follows from (10) that

$$T_{\text{cr}}(\infty) = T_{\text{cr}}(0) \left(1 - \sum_{i=1}^{3} \frac{a_i}{\beta_i}\right). \tag{11}$$

From (11) we conclude that $T_{\text{cr}}(\infty) < T_{\text{cr}}(0)$. The ratio $T_{\text{cr}}(\infty)/T_{\text{cr}}(0)$ depends on the rheological properties of the material and the geometric characteristics of the plate under investigation.

The use of the Mittag-Leffler function makes it possible to establish an exact relation between the instantaneous and the steady-state values of the critical force. Since the series (9), which defines the Mittag-Leffler function, converges slowly, for its practical application at various fixed moments of time $0 < t < \infty$ we shall use the approximation of M. I. Rozovskii ([5]), $E_\mu(-\xi) \approx e^{-\gamma_1 \xi}$, where $\gamma_1 = (1-\alpha)^{1-\alpha}$. Thus, the final value of the critical force has the form

$$T_{\text{cr}}(t) \approx T_{\text{cr}}(0) \left\{ 1 - \sum_{i=1}^{3} \frac{a_i}{\beta_i} \left[1 - \exp(\gamma_1 \beta_i t)^{1-\alpha} \right] \right\}. \tag{12}$$

Putting $\alpha = 0$, we obtain the results that follow from the theory of an elastic-viscous body in the analytical treatment of A. Yu. Ishlinskii ([6]) and A. R. Rzhanitsyn ([7]).

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Note: Figure translations are in progress. See original paper for figures.