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Abstract

Full Text

MATHEMATICS

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ON GROUPS WITH FINITELY MANY CLASSES OF CONJUGATE SUBGROUPS WITH A GIVEN PROPERTY

(Presented by Academician A. I. Mal'cev, 9 XI 1960)

The present note is devoted mainly to the study of groups in which, for every infinite subgroup, there exists a finite number of subgroups conjugate to it*. The results obtained here supplement and continue the investigations (1-3).

I. For the formulation of Theorems 1 and 5 we shall need the following definition: an infinite group every proper subgroup of which is finite will be called **quasifinite** (whether such a group is quasicyclic is the content of a well-known and still unsolved problem of O. Yu. Schmidt).

Theorem 1. *An arbitrary periodic group \mathfrak{G} with finitely many classes of conjugate infinite subgroups, containing no quasifinite subgroup, is a finite extension of its center.*

Corollary. *If in a periodic group \mathfrak{G} , containing no quasifinite subgroup, the classes of conjugate infinite subgroups are finite, then all classes of conjugate subgroups in it are finite.*

Remark 1. The requirement that the group \mathfrak{G} be periodic in Theorem 1 is essential. Indeed, consider an extension of an infinite cyclic group \mathfrak{A} by means of an automorphism group of order two, the nonidentity element of which sends every element of \mathfrak{A} to its inverse. The resulting group satisfies all the requirements of the theorem (except periodicity), but its center is equal to 1.

Remark 2. There exist periodic groups without center (even soluble ones) in which, for every infinite subgroup, there exists a finite number of subgroups conjugate to it. An example is the group which is an extension of the quasicyclic group \mathfrak{P} for a prime number $p \neq 2$ by means of an automorphism group of order two, the nonidentity element of which sends every element of \mathfrak{P} to its inverse.

Remark 3. The requirement of finiteness of all classes of conjugate **infinite** subgroups in Theorem 1 cannot be replaced by the requirement of finiteness only of the classes of conjugate **infinite abelian** subgroups. However, if the group \mathfrak{G} is assumed locally nilpotent, then the replacement can be made.

Theorem 2. *If, in an arbitrary group \mathfrak{G} with finitely many classes of conjugate infinite subgroups, the center is infinite, then its index in \mathfrak{G} is finite.*

It is known that for many classes of groups the following is true:

Theorem A. *A group satisfying the minimal condition for abelian subgroups is extremal**.*

* S. N. Chernikov drew my attention to the advisability of studying this class of groups.

** An **extremal group** is a group that is a finite extension of an abelian subgroup satisfying the minimal condition.

If a group \mathfrak{G} belongs to one of the classes in which Theorem A holds, then we shall call it an *E*-group. (In particular, the *E*-groups will include locally soluble groups, locally normal groups, etc.)

Theorem 3. *Let \mathfrak{G} be a periodic *E*-group that determines, for each of its infinite subgroups, a finite number of subgroups conjugate to it. If the center of the group \mathfrak{G} is finite, then it is extremal.*

Corollary. *Let \mathfrak{G} be a periodic *E*-group with finite classes of conjugate infinite subgroups. If the group \mathfrak{G} is not a finite extension of its center, then it is extremal.*

Remark. In Theorem 3, the condition that the group \mathfrak{G} be an *E*-group is dictated by the unsolved nature of the above-mentioned problem of O. Yu. Schmidt.

II. Let \mathfrak{G} be an arbitrary group and \mathfrak{A} some finite subgroup of it. If every abelian subgroup of the group \mathfrak{A} determines in \mathfrak{G} a finite number of subgroups conjugate to it, then, obviously, the index of the centralizer of the subgroup \mathfrak{A} is finite in \mathfrak{G} . Under the assumption that the group \mathfrak{G} is periodic and that the set $\pi(\mathfrak{A})^*$ is finite, the assertion remains valid also in the case when the subgroup \mathfrak{A} is infinite.

Theorem 4. *If \mathfrak{G} is an arbitrary periodic group and \mathfrak{A} is some subgroup of it with finite set $\pi(\mathfrak{A})$, then the index of the centralizer of the subgroup \mathfrak{A} is finite in \mathfrak{G} if and only if every abelian subgroup of the group \mathfrak{A} determines in \mathfrak{G} a finite number of subgroups conjugate to it.*

For locally normal groups this proposition was proved by the author in paper ⁽³⁾.

Corollary. *If \mathfrak{A} is a subgroup of a periodic group \mathfrak{G} and the set $\pi(\mathfrak{A})$ is finite, then from the finiteness of the classes of abelian subgroups of the group \mathfrak{A} conjugate in \mathfrak{G} there follows the finiteness of the number of subgroups of the group \mathfrak{G} conjugate to \mathfrak{A} .*

Remark 1. In Theorem 4 one cannot dispense with the requirement that the set $\pi(\mathfrak{A})$ be finite (see the example to Theorem 1 in paper ⁽³⁾).

Remark 2. Theorem 4 does not extend to arbitrary groups. An example may be furnished by a group \mathfrak{G} that is an extension of a quasicyclic group \mathfrak{A} by means of an infinite cyclic group of automorphisms.

Theorem 5. *Let \mathfrak{G} be an arbitrary periodic group and \mathfrak{A} some subgroup of it which is an E -group and for which the set $\pi(\mathfrak{A})$ is finite. If every infinite abelian subgroup of the group \mathfrak{A} determines in \mathfrak{G} a finite number of subgroups conjugate to it, then in \mathfrak{A} there exists a subgroup \mathfrak{B} of finite index whose centralizer has finite index in \mathfrak{G} .*

Theorem 6. *Let \mathfrak{G} be a periodic group and \mathfrak{A} some subgroup of it with finite set $\pi(\mathfrak{A})$, containing no quasifinite group. If every infinite subgroup of the group \mathfrak{A} determines in \mathfrak{G} a finite number of subgroups conjugate to it, then the index of the centralizer of the subgroup \mathfrak{A} is finite in \mathfrak{G} .*

Remark. From Theorem 6 there follows the assertion of Theorem 1 for the case when the set $\pi(\mathfrak{G})$ is finite.

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CITED LITERATURE

- ¹ B. H. Neumann, *Math. Zs.*, **63**, 76 (1955).
- ² S. N. Chernikov, *DAN*, **114**, 1177 (1957).
- ³ I. I. Eremin, *Matem. sborn.*, **47**, 45 (1959).

* By $\pi(\mathfrak{A})$ we understand the set of prime divisors of the orders of elements of the group \mathfrak{A} .

Note: Figure translations are in progress. See original paper for figures.

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