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Abstract

Full Text

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THE MOROZOV-BOREL THEOREM FOR REAL LIE GROUPS

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V. V. Morozov ([1]) proved that the maximal solvable subalgebras of a semisimple complex Lie algebra are conjugate with respect to the inner automorphisms of this algebra. It follows immediately from this that the maximal connected solvable subgroups of an arbitrary complex Lie group are conjugate with respect to the inner automorphisms of this group. An analogous result for algebraic linear groups over an algebraically closed field was obtained by A. Borel ([2]). In the complex case Borel' s result is identical with Morozov' s theorem, but Borel' s geometric proof is elegant.

The principal aim of the present note is to prove an analogue of the Morozov-Borel theorem for real linear Lie groups (Theorem 2). At present all maximal solvable subalgebras of real semisimple Lie algebras have been found, up to conjugacy ([3,4]). Theorem 2 could probably have been obtained on the basis of the cited works by an enumeration of cases, but it seems to us that the simple general proof given below is preferable.

1. Triangular groups

Let V be an n-dimensional real vector space.

Definition. A linear Lie group $\mathfrak{G}$ (respectively, a linear Lie algebra T) acting in the space V is called **triangular** if, in some basis of the space V, all elements of the group $\mathfrak{G}$ (respectively, of the algebra T) are represented by upper triangular matrices. It is obvious that a connected linear Lie group is triangular if and only if its Lie algebra is triangular.

Proposition 1. *A connected triangular group is closed in the full linear group.*

First of all, note that a connected triangular group is always simply connected, since it contains no nontrivial compact subgroup. Let $\mathfrak{G}$ be a connected triangular group, $\overline{\mathfrak{G}}$ its closure. The group $\overline{\mathfrak{G}}$ is also triangular and hence simply connected. Its normal divisor $\mathfrak{G}$ must be closed (see, for example, ([5]), Theorem 97). This shows that $\mathfrak{G} = \overline{\mathfrak{G}}$.

Proposition 2. *In order that a connected linear Lie group $\mathfrak{G}$ (respectively, a linear Lie algebra T) be triangular, it is necessary and sufficient that all eigenvalues of the operators from $\mathfrak{G}$ (respectively, from T) be real.*

It is enough to show that every group $\mathfrak{G}$ satisfying the formulated condition is solvable. If this were not so, then it would contain a nontrivial semisimple subgroup. Since every semisimple linear group not reducing to the identity contains a nontrivial compact subgroup, the group $\mathfrak{G}$, under the assumption made, would also contain a nontrivial compact subgroup, which is impossible.

Proposition 3. *Let G be a semisimple linear Lie algebra. In order that an operator $X \in G$ have only real eigenvalues, it is necessary and sufficient that the operator $\operatorname{ad} X : Y \to [X, Y]$ $(Y \in G)$ in the space G have this property.*

Each eigenvalue of the operator $\operatorname{ad} X$ has the form $\lambda - \mu$, where λ and μ are eigenvalues of the operator X. Therefore the proposition is valid in one direction, even independently of the semisimplicity of the algebra G.

Let the algebra G be semisimple and let the eigenvalues of the operator $\operatorname{ad} X$ be real. It suffices to consider the case when the algebra G is irreducible in the space V. Denote by A the associative algebra of operators generated by G, and put

$$\operatorname{Ad} X \cdot Y = [X, Y]$$

for all $X \in G$, $Y \in A$. Clearly, $\operatorname{Ad} X$ is a derivation of the algebra A, extending the endomorphism $\operatorname{ad} X$ of the space $G \subset A$. The eigenvalues of the operator $\operatorname{Ad} X$ are sums of eigenvalues of the operator $\operatorname{ad} X$, and hence are real. The algebra A may be either the algebra of all linear transformations of the space V, or the algebra of transformations preserving some complex or quaternionic structure in the space V. In all cases, the differences of the eigenvalues of the operator X will be eigenvalues of the operator $\operatorname{Ad} X$, and consequently real numbers. Since, moreover, the sum of all eigenvalues of the operator X, i.e. its trace, is equal to 0, they are real.

2. Lemma on the limit of a trajectory. *Let A be an endomorphism of the space V, all of whose eigenvalues are real; let PV be the projective space associated with V (its points are the lines of the space V). Every trajectory of the one-parameter group $\exp tA$ in the space PV has a limit as $t \to +\infty$.*

We agree, for every nonzero vector $a \in V$, to denote by $\bar{a}$ the corresponding point of the space PV. In a fixed basis of the space V, the matrix elements of the operator $\exp tA$ are functions of t of the form

$$\sum_{\lambda} Q_{\lambda}(t) e^{\lambda t},$$

where λ are the eigenvalues of the endomorphism A, and Q_{λ} are polynomials (see, for example, ([6])). Consequently, the coordinates of the vector

$$\exp tA \cdot x = x(t) \quad (x \in V)$$

have the same form. If $t^M e^{\lambda t}$ is the term of highest order of growth as $t \to +\infty$ occurring in the coordinates of the vector $x(t)$, then

$$x(t) = t^M e^{\lambda t}(a + \varepsilon(t)),$$

where a is a constant vector, not equal to 0, and

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

It is clear that

$$\lim_{t\to+\infty} \overline{x(t)} = \bar{a}.$$

We note that *the limit of a trajectory of a one-parameter group* of transformations of some variety *is a fixed point of this group.*

3. The space of flags. A *flag* is a collection

$$f = \{V_i,\ 1 \le i \le n\}$$

of subspaces of the space V such that

$$\dim V_i = i, \qquad V_i \subset V_{i+1}.$$

With each flag $f = \{V_i\}$ there is associated an ordered basis $\{e_i^f\}$ of the space V, for which $e_i^f \in V_i$. This basis is determined up to triangular transformations.

Every automorphism of the space V induces a certain transformation of the space of flags $\mathfrak{F}$. The group of transformations preserving the flag f consists of those transformations which, in the basis $\{e_i^f\}$, are written as upper triangular matrices.

Denote by V^k $(1 \le k \le n)$ the space of skew-symmetric k-vectors over the space V, and by PV^k the projective space associated with V^k. The space of flags $\mathfrak{F}$ is naturally identified with a closed subset of the product

$$PV^1 \times \cdots \times PV^n,$$

where the action of the linear group of the space V on $\mathfrak{F}$ is induced by the canonical linear representations of this group in the spaces V^k.

If the operator A has only real eigenvalues in the space V, then it has only real eigenvalues also in any of the spaces V^k. Consequently, the lemma on the limit of an orbit proved above remains valid when the space PV is replaced by the space $\mathfrak{F}$.

Proposition 4. A connected triangular group $\mathfrak{T}$ has a fixed point in any invariant closed subset $\mathfrak{D}$ of the space of flags.

If the group $\mathfrak{T}$ does not reduce to the identity, then it contains a normal divisor $\mathfrak{T}_1$ of dimension 1 and a one-parameter subgroup $\mathfrak{U}$ such that $\mathfrak{T} = \mathfrak{T}_1\mathfrak{U}$. The set $\mathfrak{D}_1$ of all fixed points of the group $\mathfrak{T}_1$ in $\mathfrak{D}$ is closed and invariant with respect to the group $\mathfrak{T}$. We may assume that Proposition 4 is true for the group $\mathfrak{T}_1$. Then the set $\mathfrak{D}_1$ is nonempty. Let c be the limit of some orbit of the group $\mathfrak{U}$ lying in $\mathfrak{D}_1$. It is obvious that c is a fixed point of the group $\mathfrak{T}$.

4. Decomposable groups.

Definition. A connected linear Lie group $\mathfrak{G}$ (respectively, a linear Lie algebra G) is called **decomposable** if it can be represented in the form $\mathfrak{G} = \mathfrak{K}\mathfrak{T}$ (respectively, $G = K + T$), where $\mathfrak{K}$ is a connected compact subgroup (respectively, K is the Lie algebra of a compact linear group), and $\mathfrak{T}$ is a connected triangular subgroup (respectively, T is a triangular subalgebra). Such a representation of the group $\mathfrak{G}$ (respectively, of the algebra G) is called a **polar decomposition**.

Proposition 5. A connected linear Lie group is decomposable if and only if its Lie algebra is decomposable. A decomposable group is closed in the full linear group.

Both assertions follow from the following fact: if $\mathfrak{B}$ is a closed subset and $\mathfrak{C}$ is a compact subset of a topological group $\mathfrak{A}$, then the set $\mathfrak{C}\mathfrak{B}$ is closed in $\mathfrak{A}$.

It is known (see, for example, ([7])) that the adjoint group of a semisimple Lie group is decomposable. On the other hand, Propositions 2 and 3 show that decomposability of a semisimple Lie algebra does not depend on its linear realization. Thus, every semisimple linear Lie algebra is decomposable.

Theorem 1. The identity component of an algebraic linear group (respectively, an algebraic linear Lie algebra) is decomposable.

First we shall prove that every completely reducible abelian algebraic algebra A is decomposable. In the complexification of the space V, in a certain basis consisting of real and pairs of conjugate complex vectors, the operators from A are written as diagonal matrices. Moreover, the matrices belonging to operators from A are distinguished among all diagonal matrices by some number of integer linear relations between the diagonal elements ([8]) and by conjugacy conditions on diagonal elements corresponding to conjugate basis vectors. Therefore the real and imaginary parts of each matrix of an operator from A are also matrices of operators from A. If K_A is the space of all operators from A with purely imaginary eigenvalues, and T_A is the space with real eigenvalues, then $A = K_A + T_A$, which gives a polar decomposition of the algebra A.

Now let us prove that every solvable algebraic algebra R is decomposable. If N is the ideal of the algebra R consisting of all nilpotent endomorphisms in R, then $R = N + A$, where A is a completely reducible abelian algebraic algebra ([9]). If $A = K_A + T_A$ is a polar decomposition of the algebra A, then $R = K_A + (T_A + N)$ is a polar decomposition of the algebra R.

Finally, consider an arbitrary algebraic algebra G. Its radical R is also algebraic ([9]). Let $R = K_R + T_R$ be its polar decomposition, and let C be the centralizer of the subalgebra K_R in G. Since the adjoint representation of K_R in G is completely reducible (the corresponding group

compact!) and $[K_R, G] \subset R$, then $C + R = G$. This shows that the semisimple subalgebra S of the algebra G, complementary to R, can be chosen to lie in

C. If $S = K_S + T_S$ is the polar decomposition of the algebra S, then $G = (K_S + K_R) + (T_S + T_R)$ is the polar decomposition of the algebra G.

5. The conjugacy theorem

Theorem 2. *The maximal connected triangular subgroups of an arbitrary linear Lie group* $\mathfrak{G}$ *are conjugate to one another with respect to inner automorphisms of the group* $\mathfrak{G}$.

Let the group $\mathfrak{G}$ be decomposable: $\mathfrak{G} = \mathfrak{K}\mathfrak{T}$. Then $\mathfrak{T}$ is a maximal connected triangular subgroup; it has a fixed point f in the flag space $\mathfrak{F}$. The orbit $\mathfrak{D}$ of the group $\mathfrak{G}$ passing through f is closed in $\mathfrak{F}$ as the continuous image of the compact $\mathfrak{G}/\mathfrak{T} \simeq \mathfrak{K}$. If $\mathfrak{T}'$ is any maximal connected triangular subgroup of the group $\mathfrak{G}$, then, by Proposition 4, it has a fixed point in $\mathfrak{D}$ and, consequently, is conjugate to the subgroup $\mathfrak{T}$.

In the general case we carry out the proof in terms of Lie algebras. Let G be a linear Lie algebra, and let $G = R + S$ be its Levi decomposition. The elements of the algebra R all of whose eigenvalues are real form a subspace T_R. We have $T_R \supset [G, R]$, since the elements of $[G, R]$ are nilpotent ([9]). In particular, T_R is an ideal in G. Hence it follows that every maximal triangular subalgebra T of the algebra G must contain T_R. Next, let A be a subspace in R, complementary to T_R and invariant with respect to S: $[S, A] \subset A$. Since, on the other hand, $[S, A] \subset T_R$, we have $[S, A] = 0$. Let $t = t_R + a + s$ ($t_R \in T_R,\ a \in A,\ s \in S$) be an arbitrary element of T. Then $t - t_R = a + s \in T$. The operator $\mathrm{ad}(a + s)$ has only real eigenvalues. Since $\mathrm{ad}_S\, s = \mathrm{ad}_S(a + s)$, by Proposition 3 the operator s also has only real eigenvalues. Since $[a, s] = 0$, the eigenvalues of the operator a are also real, i.e. $a \in T_R$. Consequently, $a = 0$. This shows that $T \subset T_R + S$. It is now clear that $T = T_R + T_S$, where T_S is a maximal triangular subalgebra of the algebra S. The subalgebra T_S is determined up to inner automorphisms of the algebra S, whence the assertion of the theorem follows.

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Note: Figure translations are in progress. See original paper for figures.

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